INIZIO ORE 14.10

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A APERTO, $\underline{x} \in A$, $f \in C^{(2)}_A$

MAIN CONSTRUCTION SIA $v : \|v\| = 1$ SIA

$r_{\underline{x},v} :]-\delta, \delta[\subseteq \mathbb{R} \to \mathbb{R}^n$ t.c. $r_{\underline{x},v} : t \to \underline{x} + tv$

SIA

$F_{\underline{x},v} = f \circ r_{\underline{x},v}$ CIOÈ

$F_{\underline{x},v}(t) = (f \circ r_{\underline{x},v})(t) = f(r_{\underline{x},v}(t)) = f(\underline{x} + tv)$!!!

ORA $\to (r'_{\underline{x},v,1}(t), \dots, r'_{\underline{x},v,n}(t)) = (v_1, v_2, \dots, v_n) = v$!!!

$\exists \quad F'_{\underline{x},v}(t) \overset{THM}{=} \langle \text{grad} f(\underline{x} + tv), v \rangle$

THM SE $\underline{x}$ MAX(MIN) LOCALE PER $f \Rightarrow$

$\Rightarrow 0$ MAX (MIN) LOC PER $F_{\underline{x},v} \Rightarrow$

$\Rightarrow F'_{\underline{x},v}(0) = 0$
$= \langle \text{grad} f(\underline{x}), v \rangle$ $\Big\} \Rightarrow \forall v : \|v\| = 1$

$\Rightarrow \text{grad} f(\underline{x}) = \underline{0} \in \mathbb{R}^n \Rightarrow df(\underline{x}) = \underline{0}$ CIOÈ $\underline{x}$ È PTO CRITICO

SIA $\underline{x}$ CRITICO PER f. E SIA $\underline{x}$ MAX (MIN)

$\Rightarrow 0$ MAX (MIN) PER $F_{\underline{x},v} \Rightarrow$

$F''_{\underline{x},v}(0) \leq 0$ MAX !!!
≥ 0 MIN

ORA CHI È $F''_{\underline{x},v}(0)$???

$t \in]-\delta, \delta[\subseteq \mathbb{R}$

$$F''_{\underline{x},v}(t) \overset{DEF}{=} \left(F'_{\underline{x},v}(t)\right)' =$$

$$\overset{TUM}{=} \left(\langle \operatorname{grad} f(\underline{x}+tv), v \rangle \right)' =$$

$$= \left(\sum_{i=1}^{n} \underline{\left(\frac{\partial f}{\partial x_i}(\underline{x}+t) \right)'} \cdot v_i \right) =$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i}(\underline{x}+tv) \right) v_j \right) v_i =$$

$$= \sum_{i,j=1}^{n} \frac{\partial^{(2)} f}{\partial x_j \partial x_i}(\underline{x}+tv)\, v_j \cdot v_i \,.$$

PERCIÒ

$$F''_{\underline{x},v}(0) = \sum_{i,j=1}^{n} \frac{\partial^{(2)} f(\underline{x})}{\partial x_j \partial x_i} \cdot v_j \cdot v_i \qquad (*) !!!$$

RICORDIAMO CHE LA MATRICE HESSIANA DI f IN $\underline{x}$ È:

$$H_{f(\underline{x})} = \left[\frac{\partial^{(2)} f(\underline{x})}{\partial x_i \partial x_j} \right]_{i,j=1,2,\dots,n}$$

QUADRATA DI ORDINE

$n \times n$

$+$

SIMMETRICA

$\Downarrow$

DIAGONALIZZABILE !!!

ORA, CALCOLIAMO

$$v \times H_{f(\underline{x})} \overset{DEF}{=}$$

j-ESIMA COLONNA

$$(v_1, v_2, \dots, v_i, \dots, v_n) \times \begin{pmatrix} & \frac{\partial^{(2)} f(\underline{x})}{\partial x_1 \partial x_j} & \\ & \frac{\partial^2 f(\underline{x})}{\partial x_2 \partial x_j} & \\ & \vdots & \end{pmatrix} \overset{?}{=}$$

$(1 \times n) \times (n \times n)$ IL RISULTATO HA ORDINE $1 \times n$ (VETTORE RIGA)

$$= \left(\;, \ldots , \sum_{i=1}^{n} \frac{\partial^{(2)} f(\underline{x})}{\partial x_i \, \partial x_j} v_i , \ldots \right) \text{!!!}$$

↑ POS j-ESIMA

PERCIÒ

$$\langle v \times H_{f(\underline{x})} , v \rangle =$$

$$\left\langle \left(\ldots , \sum_{i=1}^{n} \frac{\partial^{(2)} f(\underline{x})}{\partial x_i \partial x_j} v_i , \ldots \right), (v_1, \ldots, v_j, \ldots, v_n) \right\rangle =$$

$$= \sum_{j=1}^{n} \frac{\partial^{(2)} f(\underline{x})}{\partial x_i \partial x_j} v_i \cdot v_j \overset{!!!}{=} F''_{\underline{x},v}(0) \quad !!!$$

RICAPITOLANDO:

$\underline{x}$ MAX (MIN) PER $f \implies$

$\implies F''_{\underline{x},v}(0) \le 0$ MAX

$(\ge 0$ MIN $)$

$$\langle v \times H_{f(\underline{x})} , v \rangle \le 0 \text{ MAX } (1) \quad \forall v : \|v\| = 1 \quad !!!$$

$(\ge 0$ MIN$)$ (2)

PER

(1) $\iff$ $H_{f}(\underline{x})$ SEMIDEFINITA NEGATIVA $\iff$ AUTOVALORI SONO ≤ 0

(2) $\overset{\text{DEF}}{\iff}$ $H_{f}(\underline{x})$ SEMIDEFINITA POSITIVA $\overset{\text{ALG.LIN}}{\iff}$ TUTTI GLI AUTOVALORI SONO ≥ 0.

COROLLARIO SIA $\underline{x}$ CRITICO ($\nabla f(\underline{x}) = \underline{0}$)

E SIA $H_{f}(\underline{x})$ NON SEMIDEFINITA (NÉ POS, NÉ NEG).

$\Longrightarrow$ $\underline{x}$ È DI SELLA !!!