Wednesday, October 5, 2022 12:29 PM

TEOREMA FONDAMENTALE

V VARIETÀ $V \subseteq \mathbb{R}^n$ DI DIMENSIONE r ESISTE

$\forall \alpha \in V$ $\quad F : I(\alpha, \delta) \longrightarrow \mathbb{R}^{n-r}$, $F \in C^{(1)}$ LA FUNZIONE

CHE MI DESCRIVE LE EQUAZIONI.

SIA $dF(\alpha) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-r}$ LINEARE IL SUO DIFFERENZIALE IN $\alpha \in V$

$$\mathrm{Ker}\,(dF(\alpha)) \overset{DEF}{=} \{ h \in \mathbb{R}^n ;\ dF(\alpha)(h) = \underline{0} \in \mathbb{R}^{n-r} \}$$

TEOR. FOND (VERS 1)

$$T(\alpha) = \mathrm{Ker}\, dF(\alpha) \qquad !!!$$

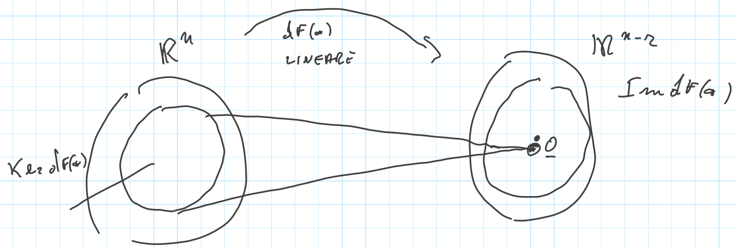
COR. 1) $T(\alpha) \overset{TFM}{=} \mathrm{Ker}\, dF(\alpha)$ È SOTTOSPAZIO VETTORIALE DI $\mathbb{R}^n$.

2) $\dim(T(\alpha)) = \dim(\mathrm{Ker}\, dF(\alpha)) = r \quad !!!$

PROOF. GRASSMANN LAWS (1844).

IN GENERALE

(1) $\quad \dim(\mathrm{Ker}\, dF(\alpha)) + \dim(\mathrm{Im}\, dF(\alpha)) = n \quad ???$



NEL NOSTRO CASO, COSA DICE

1) $\quad \dim(\mathrm{Ker}\, dF(\alpha)) + \dim(\mathrm{Im}\, dF(\alpha)) = n \quad ???$

CH'È $\mathrm{Im}\, dF(\alpha)$?? PER HP DI REGOLARITÀ:

HP REGOLARITÀ

$\dim(\operatorname{Im} dF(\alpha)) = \underset{\text{PER COLONNE}}{rk}(J_{F(\alpha)}) = n-r$

1) DIVENTA: $\dim(T(\alpha)) = \dim(\operatorname{Ker} dF(\alpha)) = n - \dim(\operatorname{Im} dF(\alpha))$

$= n - (n-r) = r$

FORMA MATRICIALE DEL TEOR. FOND.

(1) $T(\alpha) = \operatorname{Ker} dF(\alpha) \overset{DEF}{=} \{h \in \mathbb{R}^n ; dF(\alpha)(h) = \underline{0} \in \mathbb{R}^{n-r}\}$

$$(2) = \left\{ h \in \mathbb{R}^n ; J_{F(\alpha)} \times \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\} =$$

$h^t \qquad \underline{0}^t \in \mathbb{R}^{n-r}$

$$= \left\{ h \in \mathbb{R}^n ; \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\alpha) & \dots & \frac{\partial F_1}{\partial x_n}(\alpha) \\ \dots & & \dots \\ \frac{\partial F_{n-r}}{\partial x_1}(\alpha) & \dots & \frac{\partial F_{n-r}}{\partial x_n}(\alpha) \end{pmatrix} \times \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}}_{\text{in } \mathbb{R}^{n-r}} \right\}$$

$$= \left\{ h \in \mathbb{R}^n ; \begin{cases} \frac{\partial F_1}{\partial x_1}(\alpha) h_1 + \dots + \frac{\partial F_1}{\partial x_n}(\alpha) h_n = 0 \\ \dots \\ \frac{\partial F_{n-r}}{\partial x_1}(\alpha) h_1 + \dots + \frac{\partial F_{n-r}}{\partial x_n}(\alpha) h_n = 0 \end{cases} \right.$$

SIST LINEARE OMOGENEO DI $n-r$ EQS (INDIP.) IN n INCOGNITE

TEOR. FOND. VERS. 3

(3) $T(\alpha) \overset{TEOR}{=} \{ h \in \mathbb{R}^n ; \langle \operatorname{grad} F_i(\alpha), h \rangle = 0 \ \forall i = 1, 2, \dots, n-r \}$

SIA $\underset{\text{SPAZIO NORMALE}}{N(\alpha)} = (T(\alpha))^{\perp} \overset{DEF}{=} \{ v \in \mathbb{R}^n ; \langle v, h \rangle = 0 \ \forall h \in T(\alpha) \}$

$\dim(N(\alpha)) = n - \dim(T(\alpha)) = n - r$!!!

ORA (3) MI DICE CHE

$\operatorname{grad} F_1(\alpha) \quad \operatorname{grad} F_{n-r}(\alpha) \in N(\alpha)$!!!

$\operatorname{grad} F_1(\alpha), \dots, \operatorname{grad} F_{n-r}(\alpha) \dots$

= righe della Jacobiana — sono lin. indipendenti!!!

MA $\dim(N(\alpha)) = n - r$ +

PERCIÒ : THM $\{\operatorname{grad} F_1(\alpha), \dots \operatorname{grad} F_{n-r}(\alpha)\}$ È BASE

PER LO SPAZIO NORMALE $N(\alpha)$!!!

OTTIMIZZAZIONE CON VINCOLO:

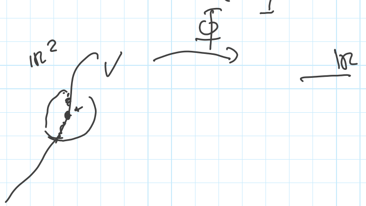
$\Phi : A \subseteq \mathbb{R}^n \to \mathbb{R}$ APERTO

SIA V VARIETÀ REGOLARE DI DIM r DI $\mathbb{R}^n$ (VINCOLO)

SIA $\alpha \in V$. DIREMO CHE α MAX (MIN) VINCOLATO

RISPETTO A $V \overset{\text{DEF}}{\iff} \exists\, I(\alpha, \delta)$ t.c.

$$\Phi(\alpha) \geq \Phi(x) \quad \forall x \in I(\alpha,\delta) \cap V.$$
$$(\leq \Phi(x))$$



TEOR. DI LAGRANGE

SE $\alpha \in V$ MAX (MIN) VINCOLATO RISPETTO A V PER Φ

$$\Rightarrow \operatorname{grad} \Phi(\alpha) \in N(\alpha) \qquad (*)$$

CLASSICO — RICORDO CHE

$\{\operatorname{grad} F_1(\alpha), \dots \operatorname{grad} F_{n-r}(\alpha)\}$ È BASE PER $N(\alpha)$!!!

ALLORA: — MOLTIPLICATORI (COEFFICIENTI)

$$\operatorname{grad} \Phi(\alpha) \doteq \lambda_1 \operatorname{grad} F_1(\alpha) + \lambda_2 \operatorname{grad} F_2(\alpha) + \dots + \lambda_{n-r} \operatorname{grad} F_{n-r}(\alpha)$$

$\operatorname{grad} \Phi(\alpha) = \lambda_1 \operatorname{grad} F_1(\alpha) + \dots + \lambda_{n-2} \operatorname{grad} F_{n-2}(\alpha)$

$\lambda_i \in \mathbb{R}$

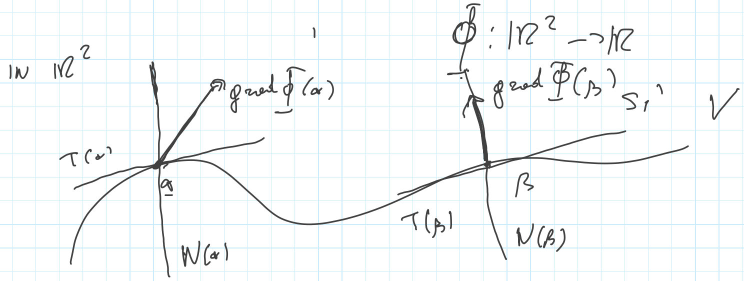
$\Updownarrow$

$\exists\, \lambda_1, \dots, \lambda_{n-2} \in \mathbb{R}$ t.c.

$$\operatorname{grad}\left(\Phi - \sum_{i=1}^{n-2} \lambda_i F_i(\alpha)\right) = 0$$

LAGRANGIANA

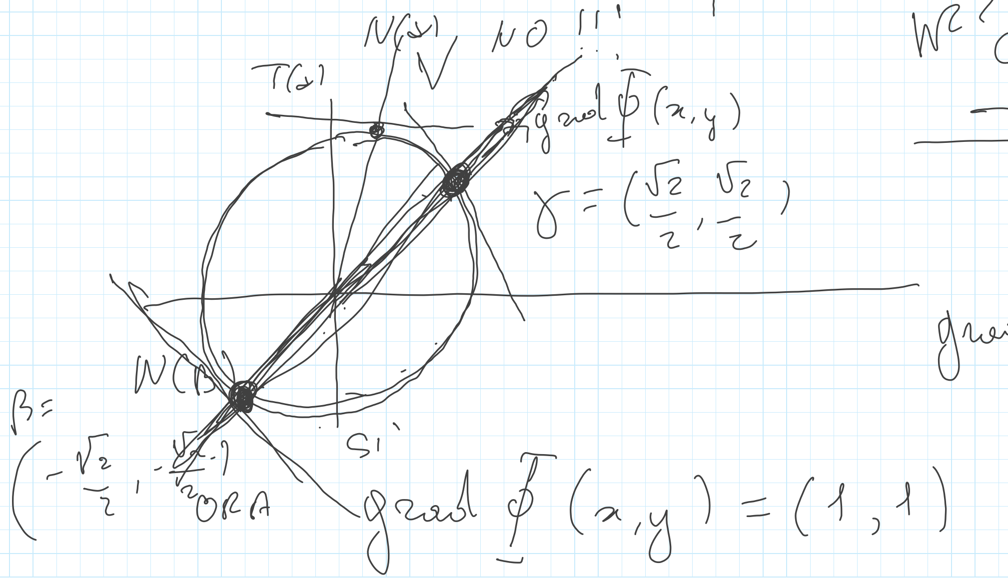
IN $\mathbb{R}^2$ $\quad \Phi : \mathbb{R}^2 \to \mathbb{R}$

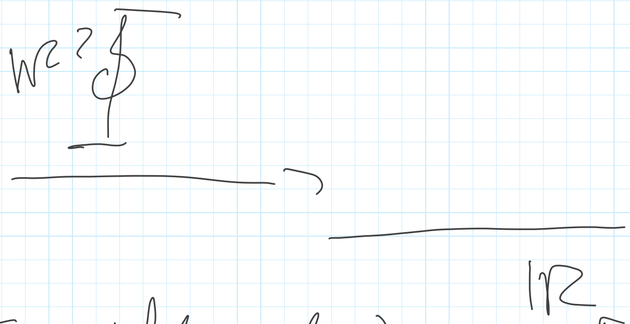


EX $\quad \Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi(x,y) = x + y$

$V = \{ (x,y) \in \mathbb{R}^2 \;;\; x^2 + y^2 = 1 \}$ (1-VARIETÀ)

CI SONO PIÙ DI MAX(MIN) VINCOLATI RISPETTO A V?





$\operatorname{grad} \Phi \notin N(\beta)$, $\operatorname{grad} \Phi \in N(\beta)$

$\operatorname{grad} \Phi(x,y) = (1, 1)$