Tuesday, November 8, 2022 11:38 AM

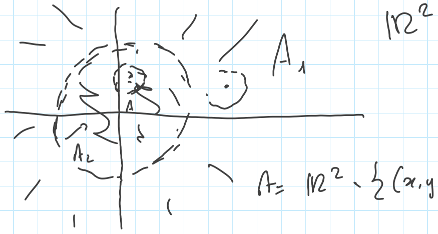
LEZ 3BIS (BEGIN AT 15.15)

CAN YOU SEE / HEAR ME?

EXAMPLE $\mathbb{R}^2$ EUCLIDEAN , "THE REAL PLANE"

AND

$$A = \{ (x,y) \in \mathbb{R}^2 \; ; \; x^2+y^2 \neq 1 \}$$



$$A = \mathbb{R}^2 \setminus \{ (x,y) \in \mathbb{R}^2 \; ; \; x^2+y^2 = 1 \}$$

IS A DISCONNECTED? YES

NOT TRIVIAL

$$A = \{ (x,y) \in \mathbb{R}^2 \; ; \; x^2+y^2 > 1 \} \,\dot{\cup}\, \{ (x,y) \in \mathbb{R}^2 \; ; \; x^2+y^2 < 1 \}$$

OPEN OPEN

A DISCONNECTED ???

THM (EUCLIDEAN CONNECTION THM)

FOR EVERY $n \in \mathbb{Z}^+$, THE EUCLIDEAN SPACE

$\mathbb{R}^n$ IS A CONNECTED SPACE !!!

RMK CONSIDER

$$\mathbb{Q} = \{ q \in \mathbb{R} \; ; \; q \text{ RATIONAL} \} \subseteq \mathbb{R}$$

CLEARLY. $\mathbb{Q}$ IS A METRIC SUBSPACE OF $\mathbb{R}$ EUCLIDEAN.

QUESTION : IS $\mathbb{Q}$ A CONNECTED SPACE? NO ???

WHY ???

CONSIDER $x_0 = \sqrt{2}$. IT IS IRRATIONAL !!!

$$x_0 := \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$$

$\mathbb{R}$

$$\mathbb{Q}_{<\sqrt{2}} \overset{\text{DEF}}{=} \{ q \in \mathbb{Q} \ ; \ q < \sqrt{2} \}$$

RATIONALS

$$\mathbb{Q}_{>\sqrt{2}} \overset{\text{DEF}}{=} \{ q \in \mathbb{Q} \ ; \ q > \sqrt{2} \}$$

CLEARLY :

$$\mathbb{Q}_{<\sqrt{2}} \cap \mathbb{Q}_{>\sqrt{2}} = \emptyset \qquad !!!$$

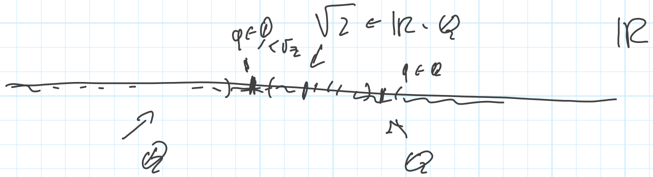
AND

$$\mathbb{Q}_{<\sqrt{2}} \,\dot{\cup}\, \mathbb{Q}_{>\sqrt{2}} = \mathbb{Q} \qquad !!!$$

CLEARLY , $\mathbb{Q}_{<\sqrt{2}} , \mathbb{Q}_{>\sqrt{2}} \neq \emptyset , \mathbb{Q}$ (NOT TRIVIAL)

FURTHERMORE :

$\mathbb{Q}_{<\sqrt{2}} , \mathbb{Q}_{>\sqrt{2}}$ ARE <u>OPEN</u> IN THE SUBSPACE $\mathbb{Q}$!!!



CONNECTION PROPERTY
AND
OPEN / CLOSED SUBSETS

$]0,1] \subseteq \mathbb{R}$ EUCLIDEAN
NOT OPEN / NOT CLOSED!

COROLLARY (RECALL) (X, d) METRIC SPACE . THEN

(X,d) CONNECTED $\overset{\text{DEF}}{\iff}$ IF A_1, A_2 ARE OPEN

AND i) $A_1 \cap A_2 = \emptyset$

ii) $A_1 \dot{\cup} A_2 = X$

THEN A_1, A_2 ARE TRIVIAL !!!

THM (X,d) CONNECTED $\overset{\text{THM}}{\iff}$ $A \subseteq X$

A OPEN/CLOSED

THEN $A = \emptyset, X$

COROLLARY (X,d) CONNECTED SPACE $\iff$

IF $A \neq \emptyset, X$ THEN

i) A OPEN $\Rightarrow$ A IS NOT CLOSED

ii) A CLOSED $\Rightarrow$ A IS NOT OPEN

PROOF OF THE THM

i) $\Rightarrow$) ASSUME THAT (X,d) CONNECTED.

BY CONTRADICTION, ASSUME THAT

(*) $\exists A \subseteq X$, $A \neq \emptyset, X$, A OPEN/CLOSED.

LET $A_1 = A$ OPEN, LET $A_2 = X \setminus A = X \setminus A_1$

WE HAVE THAT A CLOSED $\overset{\text{DEF}}{\iff}$ $A_2 = X \setminus A = A^c$ OPEN

THEN $X = \underset{\text{OPEN}}{A_1} \dot{\cup} \underset{\text{OPEN}}{A_2}$ $\qquad A_1, A_2 \neq \emptyset, X$

$\Rightarrow (X,d)$ DISCONNECTED (A CONTRADICTION)

QED

$\Leftarrow$) ASSUME THAT

$\exists A \subseteq X$, A NON TRIVIAL , A OPEN/CLOSED

$\Longrightarrow (X, d)$ DISCONNECTED SPACE

STOP ? QUESTIONS?

BYEBYE