Tuesday, November 8, 2022 11:40 AM

HELLO!

BEGIN AT 14.10

AB.

CAN YOU SEE / HEAR ME?

(FEEDBACK, PLEASE...)

NORMED SPACES LET

$(X, \|\ \|)$, $\|\ \| : X \longrightarrow \mathbb{R}$ IS A NORM

VECTOR SPACE!

(AN ALGEBRAIC STRUCTURE)

$\Updownarrow$ DEF.

SATISFIES THE FOLLOWING AXIOMS!

i) $\forall x \in X$ (VECTOR SPACE)

$\|x\| \geq 0$; $\|x\| = 0 \in \mathbb{R} \iff x = \underline{\underline{0}}$ (ZERO VECTOR VECTOR IN X)

ii) $\lambda \in \mathbb{R}$, THEN $\forall x \in X$

$\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$

iii) $\forall x, y \in X$ WE HAVE:

$\|x+y\| \leq \|x\| + \|y\|$ (TRIANGULAR INEQUALITY FOR NORMS? WHY???

FUNDAMENTAL FACT

ANY NORMED SPACE IS "CANONICALLY (???)" A METRIC SPACE!

CRUCIAL FACT: GIVEN $x, y \in X$ NORMED SPACE,

SET, $d : X \times X \longrightarrow \mathbb{R}$ S.T.

$$(*) \quad d(x,y) \overset{DEF}{=} \| x - y \| . \quad \text{THEN}$$

$d(x,y) = \| x - y \|$ IS A METRIC FUNCTION.

MAIN EXAMPLE (FOR OUR COURSE)

LET $\underset{\text{VECTOR SPACE}}{X} = \mathbb{R}^n$ $\qquad (n \in \mathbb{Z}^+)$, THAT IS

$$X = \mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \; ; \; x_i \in \mathbb{R} \, , \, i = 1, 2, \ldots, n \}$$

AND

$\underline{x} = (x_1, \ldots, x_n)$, $\underline{y} = (y_1, \ldots, y_n)$ WE HAVE:

i) $\underline{x} + \underline{y} = (x_1 + y_1, \ldots, x_n + y_n)$

ii) $\lambda \in \mathbb{R}$,

$$\lambda \underline{x} = (\lambda x_1, \ldots, \lambda x_n) .$$

(AND $\underline{0}$ ZERO VECTOR, $\underline{x} = (0, \ldots, 0)$).

DEFINE

$$\| \underline{x} \| \overset{DEF}{=} \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$ IS A NORM

THAT IS CALLED THE

"EUCLIDEAN(?) NORM" ON $\mathbb{R}^n$

WHY (?)? SINCE, COND $(*)$ SPECIALIZES TO:

↓ SPECIAL CASE!

$$d(\underline{x},\underline{y}) \overset{DEF}{=} \|\underline{x}-\underline{y}\| =$$

$$= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} \leftarrow$$

THIS IS THE "EUCLIDEAN DISTANCE".

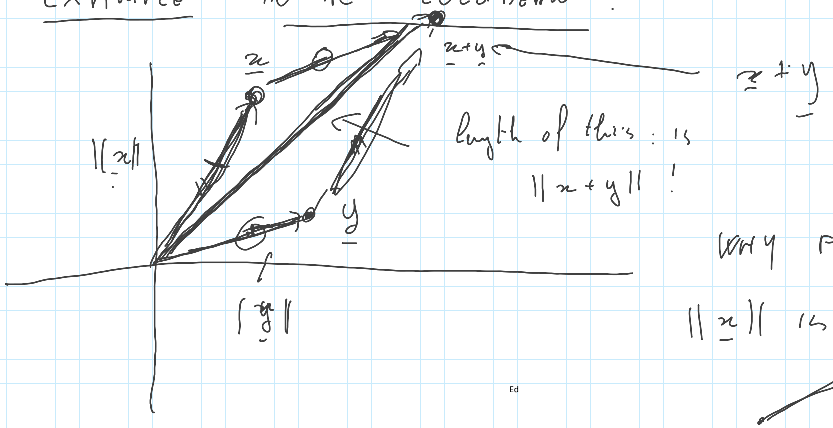
GENERAL FORM IN $\mathbb{R}^n$ THERE ARE:

"MINKOWSKY NORMS".

NOW, WHY AXIOM iii)

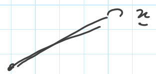
$\|x+y\| \le \|x\| + \|y\|$ IS CALLED "TRIANGULAR INEQUALITY FOR NORMS" ???

EXAMPLE IN $\mathbb{R}^2$ EUCLIDEAN.



WHY FOR ISTANCE,

$\|\underline{x}\|$ IS THE LENGTH OF



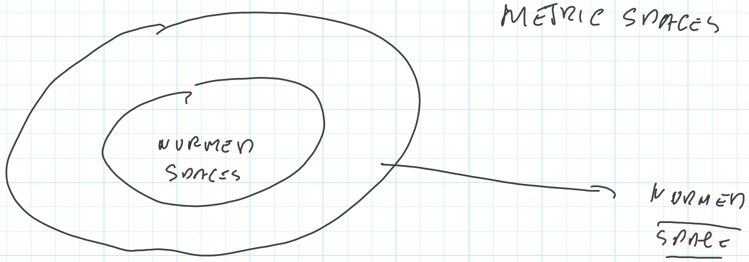
?

BUT, SINCE:

$$\|\underline{x}\| = d(\underline{x},\underline{0}) \quad !!!$$

WELL, JUST TO MAKE A (NAIVE!) PICTURE:

METRIC SPACES



WHY THE INCLUSION IS STRICT ?!?!?!

METRIC SPACES ARE JUST (ARE SETS OF POINTS) SETS ?!!?

RMK SUBSPACES. WE RECALL:

(X, d) METRIC SPACE, $A \subseteq X$ ANY SUBSET,

THEN

$(A, d_{/A})$ INHERITS A STRUCTURE OF METRIC SPACE

$a, a' \in A$, $\quad d_{/A}(a, a') \stackrel{DEF}{=} d(a, a')$ ..

LET $(X, \| \, \|)$ NORMED SPACE $\Longrightarrow$

VECTOR SPACE

IT IS A METRIC SPACE, BY SETTING.

$$d(x, y) \stackrel{DEF}{=} \|x - y\| \quad . \; !!!$$

IF $A \subseteq X$ ANY SUBSET OF X (VECTOR SPACE!),

DOES A INHERITS A STRUCTURE OF NORMED SPACE? NO.

THE ANSWER IS <u>YES</u> $\Longleftrightarrow$ $A \subseteq X$ IS A VECTOR SUBSPACE!

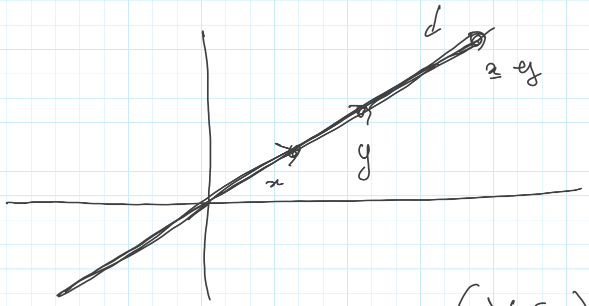
THAT IS $\forall a, a' \in A$ WE HAVE:

i) $a + a' \in A$

ii) $\lambda \in \mathbb{R}$, $\lambda a \in A$.

FOR INSTANCE, IN $\mathbb{R}^2$ EUCLIDEAN:

<u>EX 1)</u> LET A:



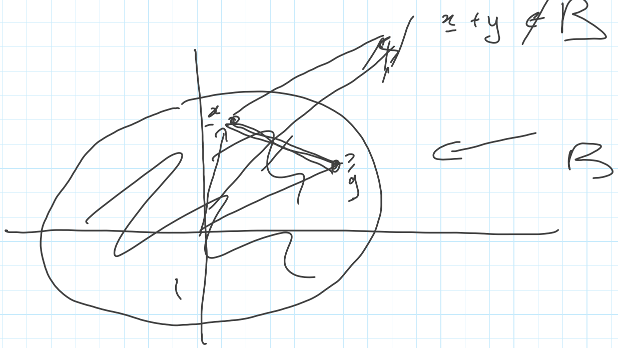
$\mathbb{R}^2$

A IS A SUBSPACE OF DIM 1:

THEN A IS <u>NORMED SPACE</u>

(<u>YES</u>)

<u>EX 2</u> IN $\mathbb{R}^2$ EUCLIDEAN:



$\mathbb{R}^2$

$\leftarrow$ B IS <u>NOT</u> A VECTOR SPACE!!!

SO, WE CANNOT SPEAK ABOUT NORM ON IT,

BUT WE CAN SPEAK ABOUT DISTANCES? YES.

BREAK QUESTIONS?

BEGIN AT 15,15.