Monday, November 21, 2022 11:07 AM

MULTIVARIATE DIFFERENTIAL CALCULUS (BEGIN)

RECALL GIVEN $f :]a,b[\subseteq \mathbb{R} \to \mathbb{R}$

$x \in]a,b[$. WE HAVE:

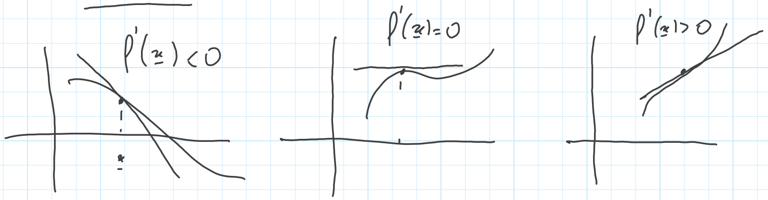
f ADMITS **DERIVATIVE** AT $x \in]a,b[$ $\overset{\text{DEF}}{\Longleftrightarrow}$

$\exists$ FINITE $\displaystyle\lim_{t \to 0} \frac{f(x+t) - f(x)}{t} \; (= f'(x) \in \mathbb{R})$

NOTATION

INCREMENTAL RATIO

WE RECALL:



DIRECTIONAL DERIVATE.

DIRECTION (OR, VERSORS) IN $\mathbb{R}^n$.

LET $v \in \mathbb{R}^n$ IS SAID TO BE A

DIRECTION (**VERSOR**) $\Longleftrightarrow \|v\| = 1$.

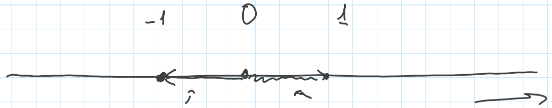
EX $n = 1$ $\mathbb{R}^1 = \mathbb{R}$

NOW, IF $n = 1$, $\|v\| = |v|$, $v \in \mathbb{R}$

$\Downarrow$

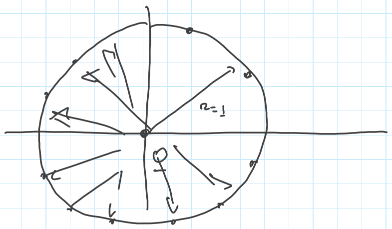
$v \in \mathbb{R}$, $\|v\| = |v| \overset{!}{=} 1 \Longrightarrow$ WE HAVE JUST TWO DIRECTIONS:

OR $v = -1$ EITHER $v = 1$ $\mathbb{R}^n$



FROM RIGHT TO LEFT FROM LEFT TO RIGHT

EX IN $n=2$???



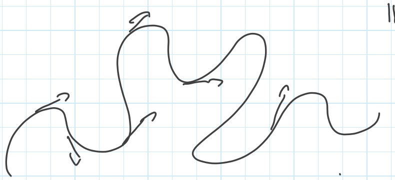
$\mathbb{R}^2$

WHICH POINTS/VECTORS ARE DIRECTIONS???

WE RECALL: v DIRECTION $\overset{DEF}{\iff}$

$$\underline{1} \overset{DEF}{=} \|v\| \overset{RECALL}{=} d(v, \underline{0}) \quad ???$$

EX 2 BIS



$\mathbb{R}^2$

DIRECTIONAL DERIVATIVE

LET $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

FIX A DIRECTION $v \in \mathbb{R}^n$, $\|v\| = 1$

WE SAY THAT:

f ADMITS DIRECTIONAL DERIVATIVE

AT THE POINT $\underline{x} \in A$ ALONG THE DIRECTION $v : \|v\| = 1$???

$\Updownarrow$ DEF

$\exists$ FINITE:

$$(*) \quad \overset{???}{=} \lim_{t \to 0 \in \mathbb{R}} \frac{f(\underline{x} + tv) - f(\underline{x})}{t} \left(= \frac{\partial f}{\partial v}(\underline{x}) \right) \in \mathbb{R}$$

NOTATION

INCREMENTAL (GENERALIZED) RATIO

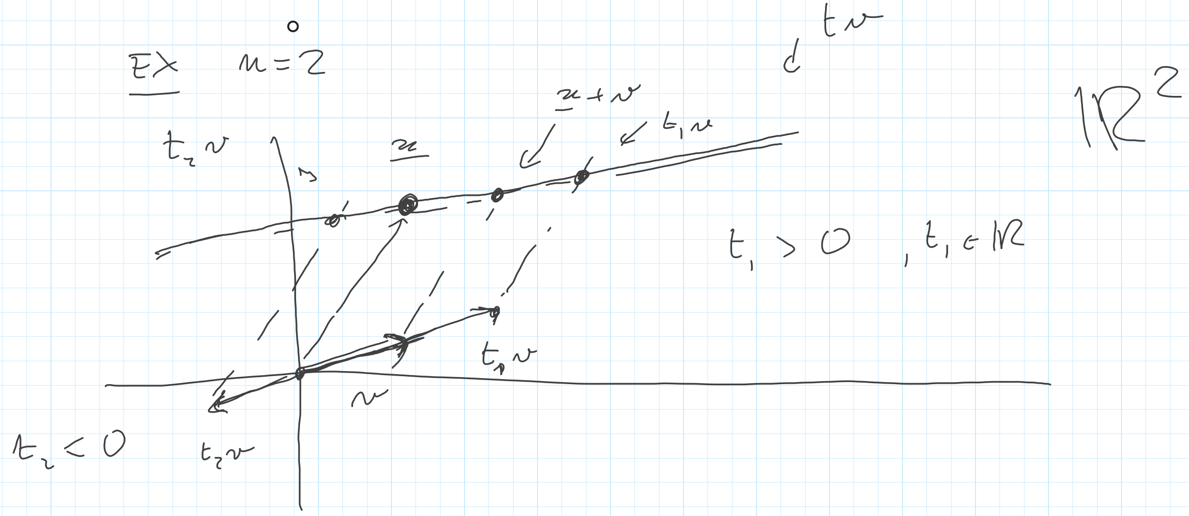
NOW, CONSIDER

$$L_v = \{ \underline{x} + tv : t \in \mathbb{R} \} \subseteq \mathbb{R}^n$$

$\ell_{x,v} = \{x + tv;\ t \in \mathbb{R}\} \subseteq \mathbb{R}^n$

WHAT KIND OF SUBSET OF $\mathbb{R}^n$ IS IT ...???

<u>EX</u> $n = 2$

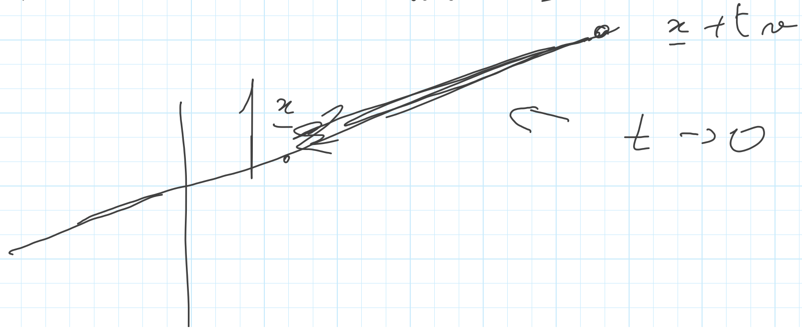


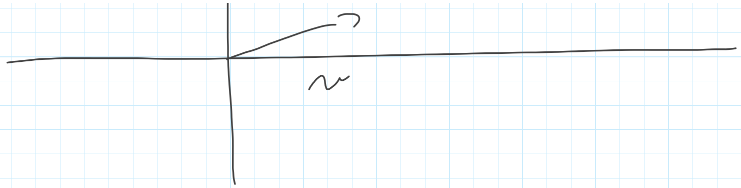
HENCE, WE DISCOVERED THAT

$\ell_{x,v} = \{x + tv;\ t \in \mathbb{R}\}$ DESCRIBES THE UNIQUE LINE PASSING THROUGH x AND HAVING DIRECTION (PARALLEL) TO v.

IN $\mathbb{R}^n$

IN PLAIN WORDS

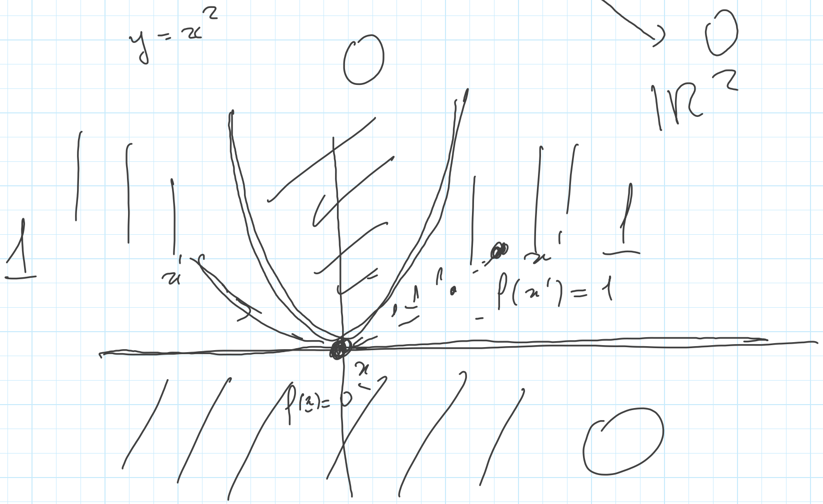


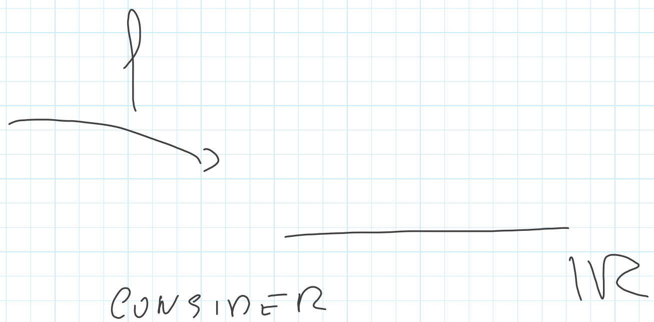


COUNTEREXAMPLE

$f : \mathbb{R}^2 \to \mathbb{R}$ WHERE

$$(x,y)\in\mathbb{R}^2 : \quad f(x,y) = \begin{cases} 0 & y \geq x^2 \\ 1 & \text{OTHERWISE} \\ 0 & y \leq 0 \end{cases}$$





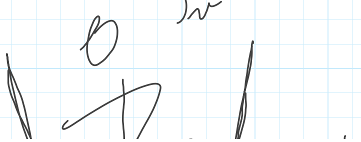
CONSIDER

$\underline{x} = (0,0)$

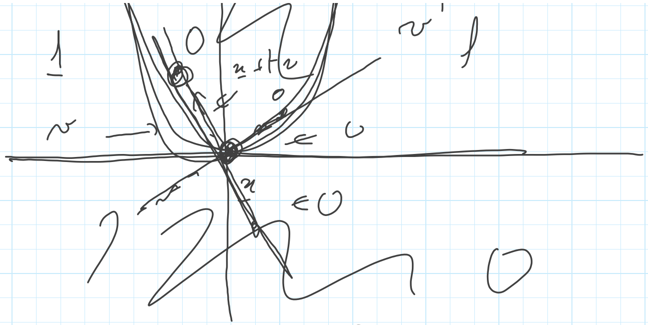
IS f CONTINUOUS AT $\underline{x} = (0,0)$??? $\underline{\underline{\text{NO}}}$!!!

BUT WHAT ABOUT DIRECTIONAL DERIVATIVES

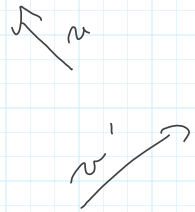
$$\frac{\partial f}{\partial v}(\underline{x}) \quad \text{AT} \quad \underline{x} = (0,0)$$



CHOOSE A DIRECTION



$\|v\| = 1$



THEN

$$\lim_{t \to 0} \frac{\overbrace{f(\underline{x}+t\underline{v})}^{=0} - \overbrace{f(\underline{x})}^{=0}}{t} \overset{!}{=} 0 \implies \exists \frac{\partial f}{\partial v}(\underline{x}) \overset{!!!}{=} 0$$

THEN, f ADMITS <u>ALL</u> DIRECTIONAL DERIVATIVES

$\frac{\partial f}{\partial v}(\underline{x})$ AT $\underline{x} = (0,0)$. FURTHERMORE:

↓ EQUALS 0 !!!

BUT <u>f NOT CONTINUOUS AT THE POINT $\underline{x} = (0,0)$!!!</u>

<u>DIFFERENTIABLE FUNCTIONS</u>

<u>STOP</u> <u>QUESTIONS</u>?

<u>BYEBYE</u> !!!