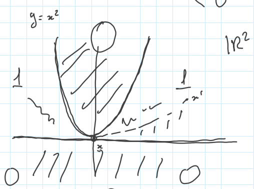
Monday, November 21, 2022 11:07 AM

DIFFERENTIABLE FUNCTS

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

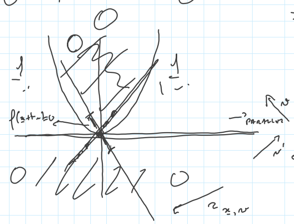
RECALL (EX OF YESTERDAY) $f : \mathbb{R}^2 \to \mathbb{R}$ S.T.

$$f(x,y) = \begin{cases} 0 & y \geq x^2 \\ 1 & \text{OTHERWISE} \\ 0 & y \leq 0 \end{cases} \qquad (x,y) \in \mathbb{R}^2$$



FIX $\underline{x} = (0,0)$ $f(\underline{x}) = f(0,0) = 0$!!!

$\Longrightarrow$ f NOT CONTINUOUS AT $\underline{x} = (0,0)$



FIX A "DIRECTION": $\|\underline{v}\| = 1$

AND CONSIDER THE LINE

$\underline{v} \in r_{\underline{x},\underline{v}} \stackrel{DEF}{=} \{ \underline{x} + t\underline{v} ;\ t \in \mathbb{R} \}$

$\exists ?$ $\dfrac{\partial f}{\partial \underline{v}}(\underline{x})$??? THIS SHOULD MEAN:

$\exists$ FINITE $\displaystyle\lim_{t \to 0 \in \mathbb{R}} \frac{\overset{=0}{f(\underline{x}+t\underline{v})} - \overset{=0}{f(\underline{x})}}{t}$? YES

FURTHERMORE $\exists \dfrac{\partial f}{\partial \underline{v}}(\underline{x}) \stackrel{!}{=} 0$

DIFFERENTIABLE FUNCTS

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

DEF f IS DIFFERENTIABLE AT $\underline{x} \in A$ $\stackrel{DEF}{\Longleftrightarrow}$

$\stackrel{DEF}{\Longleftrightarrow}$ $\exists\, L_{\underline{x}} : \mathbb{R}^n \to \mathbb{R}$ LINEAR

SUCH THAT (∗)

$$\lim_{\underline{h} \to \underline{0} \in \mathbb{R}^n} \frac{f(\underline{x}+\underline{h}) - f(\underline{x}) - L_{\underline{x}}(\underline{h})}{\|\underline{h}\|} = 0 \quad !!!$$

RECALL $L_{\underline{x}}$ LINEAR $\Longleftrightarrow$ $\begin{cases} L_{\underline{x}}(\underline{h}_1 + \underline{h}_2) = L_{\underline{x}}(\underline{h}_1) + L_{\underline{x}}(\underline{h}_2) & \text{ADDITIVITY} \\ \lambda \in \mathbb{R},\ L_{\underline{x}}(\lambda \cdot \underline{h}) = \lambda \cdot L_{\underline{x}}(\underline{h}) & \text{HOMOGENEITY} \end{cases}$

A NATURAL PRELIMINARY QUESTION: WHY, IF $n = 1$,

"NO DIFFERENTIABLE FUNCTS" ?

PROP LET $n = 1$. THEN

f DIFFERENTIABLE AT $\underline{x} \in A \iff f$ ADMITS DERIVATIVE $f'(\underline{x}) \in \mathbb{R}$ AT $\underline{x} \in A$.

PROOF $\Rightarrow$) f DIFF. AT $\underline{x} \in A \overset{DEF}{\iff} \exists L_{\underline{x}} : \mathbb{R}^n \to \mathbb{R}$ LINEAR

S.T. $$\lim_{h \to 0 \in \mathbb{R}} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{|h|} = 0 \quad !!! \qquad (**)$$

SINCE $n = 1$, $\|h\| = |h|$

$$(**) \iff \lim_{h \to 0 \in \mathbb{R}} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{h} = 0 \iff$$

BUT $h \in \mathbb{R}$. THEN $h = h \cdot 1$

$$\iff \lim_{h \to 0 \in \mathbb{R}} \frac{f(\underline{x}+h) - f(\underline{x})}{h} = \lim_{h \to 0 \in \mathbb{R}} \frac{L_{\underline{x}}(h)}{h}$$

$L_{\underline{x}}$ LINEAR

$$= \lim_{h \to 0 \in \mathbb{R}} \frac{L_{\underline{x}}(h \cdot 1)}{h}$$

$$= \lim_{h \to 0 \in \mathbb{R}} \frac{\cancel{h} \cdot L_{\underline{x}}(1)}{\cancel{h}} = L_{\underline{x}}(1) \in \mathbb{R}$$

WE PROVED

$$\lim_{h \to 0 \in \mathbb{R}} \frac{f(\underline{x}+h) - f(\underline{x})}{h} = L_{\underline{x}}(1) \in \mathbb{R} \quad !!!$$

THEN $\exists\ f'(\underline{x}) = L_{\underline{x}}(1) \in \mathbb{R}$. (MORE)

QED

TERMINOLOGY : THE LINEAR FUNCT $L_{\underline{x}}$ IS CALLED

THE DIFFERENTIAL OF f AT $\underline{x} \in A$.

$\Leftarrow$) WHAT IS "THE CANDIDATE" FOR $L_{\underline{x}} : \mathbb{R} \to \mathbb{R}$ LINEAR ???

RMK $L : \mathbb{R} \longrightarrow \mathbb{R}$ LINEAR. WHAT IS THE "FORM" OF $L_{\underline{x}}$?

$h \in \mathbb{R}$, $h = h \cdot 1 \overset{\text{LINEARITY}}{\Longrightarrow} L(h) = L(h \cdot 1) \overset{\text{LIN}}{=} h \cdot \underbrace{L(1)}_{\text{COSTANT}}$

THEN $L : \mathbb{R} \to \mathbb{R}$ LINEAR $\iff L(h) = h \cdot \overset{\text{COST}}{L(1)} \quad h \in \mathbb{R}$!!!

$$= \underbrace{L(1)}_{\text{COST}} \cdot h$$

IN OTHER WORDS : $L : \mathbb{R} \to \mathbb{R}$ LINEAR $\iff \underline{L(x) = K \cdot x}$

SO, ASSUME (IN OUR PROOF) THAT

$L_{\underline{x}} : \mathbb{R} \to \mathbb{R}$ LINEAR IS $\underline{L_{\underline{x}}(h) = f'(\underline{x}) \cdot h}$ $\quad \underline{h \in \mathbb{R}}$!!!

NOW, IS TRUE $\quad \underline{h = h \cdot 1}$

$$(+) \quad \lim_{h \to 0} \frac{f(\underline{x}+h) - f(\underline{x}) - \widehat{L_{\underline{x}}}(h)}{|h|} \underline{\underline{=}} 0 \quad ??? \quad \text{(DIFF. COND.)} \quad \underline{\text{YES}}$$

$$(+) \iff \lim_{h \to 0} \frac{f(\underline{x}+h) - f(\underline{x}) - h \cdot f'(\underline{x})}{h} = 0 \quad ??? \quad (\#) \quad \underline{\text{YES}}$$

$$(\#) \Longleftrightarrow \lim_{h\to 0} \frac{f(\underline{x}+h)-f(\underline{x})}{h} = \lim_{h\to 0} \frac{f'(\underline{x})\cdot \cancel{h}}{\cancel{h}} = \underline{\underline{f'(\underline{x})}}$$

IS IT TRUE ???

YES, BY DEF !!!

WE PROVED :

f ADMITS DERIVATIVE $f'(\underline{x}) \Longrightarrow f$ DIFF AT $\underline{x} \in A$, PLUS

$L_{\underline{x}}(h) = f'(\underline{x})\cdot h$!!!

BREAK

QUESTIONS?

BEGIN AGAIN AT 12.15