Monday, November 21, 2022 11:08 AM

LEZ 7 BIS

RECALL f DIFF AT $\underline{x} \in A \overset{DEF}{\iff} \exists L_{\underline{x}} : \mathbb{R}^n \to \mathbb{R}$ LINEAR ← DIFFERENTIAL OF f AT $\underline{x}$

S.T. $$\lim_{h \to \underline{0} \in \mathbb{R}^n} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{\|h\|} = 0 \qquad (*) \quad ???$$

A SIMPLE EXPLANATION, IN TERMS OF "APPROXIMATION THEORY".

WRITE: $E_{\underline{x}}(h) = f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h) =$

$= f(\underline{x}+h) - \big(\underbrace{f(\underline{x})}_{\text{COST}} + L_{\underline{x}}(h)\big)$ ← LINEAR FUNCT.

"THE ERROR WE MAKE" WHEN WE APPROXIMATE THE VALUE

$f(\underline{x}+h)$ BY THE MUCH SIMPLER VALUE:

$\underbrace{f(\underline{x})}_{\text{COST}} + L_{\underline{x}}(h)$ ← LINEAR FUNCT

AND THE COND:

f DIFF AT $\underline{x} \in A \iff \lim_{h \to \underline{0} \in \mathbb{R}^n} \frac{E_{\underline{x}}(h)}{\|h\|} = 0$ STRONGER $\left(\begin{smallmatrix} \Longrightarrow \\ \not\Longleftarrow \end{smallmatrix} \lim_{h \to 0} E_{\underline{x}}(h) = 0 \right)$!!!

TEST? f LINEAR FUNCT. IS IT TRUE, THAT

f IS DIFFERENTIABLE? (AT ANY POINT?) WHAT IS $L_{\underline{x}}$?

YES, AND FURTHERMORE $L_{\underline{x}} = f$!!! LET US PROOF THIS!!!

NOW, ASSUME THAT $L_{\underline{x}} = f$. IS IT TRUE: THIS MEANS:

$$? \lim_{h \to \underline{0} \in \mathbb{R}^n} \frac{f(\underline{x}+h) - f(\underline{x}) - f(h)}{\|h\|} = 0 \quad ??? \qquad (**)$$

BUT, SINCE f LINEAR $(**)$ BECOMES:

$$(**) \lim_{h \to \underline{0} \in \mathbb{R}^n} \frac{\cancel{f(\underline{x})} + \cancel{f(h)} - \cancel{f(\underline{x})} - \cancel{f(h)}}{\|h\|} = 0 \quad !!! \quad \text{YES} \quad (\text{OK???})$$

IN GENERAL, WE HAVE:

PROP GIVEN ANY $n \in \mathbb{Z}^+$, $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

IF f DIFFERENTIABLE AT $\underline{x} \in A$,

THEN $\forall \|v\| = 1$ (DIRECTION)

THE DIRECTIONAL DERIVATIVE $\frac{\partial f}{\partial v}(\underline{x})$ EXISTS!!!

MORE $$\frac{\partial f}{\partial v}(\underline{x}) \overset{!}{=} L_{\underline{x}}(v) \quad !!!$$

PROOF HP f DIFF AT $\underline{x} \in A \overset{DEF}{\iff} \exists L_{\underline{x}} : \mathbb{R}^n \to \mathbb{R}$ LINEAR

S.T. $$\lim_{h \to \underline{0} \in \mathbb{R}^n} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{\|h\|} = 0 \qquad (*)$$

$\Downarrow$

SPECIALIZE, ASSUMING THAT $h = t \cdot v$ ← DIRECTION, $t \in \mathbb{R}$

$h = t \cdot v$ ← CUST

$\downarrow \ \downarrow$

0 DEN

SPECIAL FORM

$$(*) \Longrightarrow \quad \lim_{\substack{t v \to 0 \\ \Updownarrow \\ t \to 0 \in \mathbb{R}}} \frac{f(x+h) - f(x) - L_x(t \cdot v)}{\| t \cdot v \|} = 0 \quad \underline{\text{YES}} \quad (**)$$

$$\not\Longleftarrow$$

NOW $L_x(t \cdot v) \overset{\text{LIN}}{=} \underline{t \cdot L_x(v)}$ & $\|t \cdot v\| = |t| \cdot \underset{1}{\|v\|} = |t|$

THEN $(**)$ BECOMES

$$(***) \quad \lim_{t \to 0 \in \mathbb{R}} \frac{f(x+tv) - f(x) - t \cdot L_x(v)}{|t|} \underline{= 0}$$

$$\Updownarrow$$

$$\lim_{t \to 0 \in \mathbb{R}} \frac{f(x+tv) - f(x) - t L_x(v)}{t} = 0$$

$$\Updownarrow$$

$$\underline{\lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}} = \lim_{t \to 0} \frac{\cancel{t} \cdot L_x(v)}{\cancel{t}} = \underline{L_x(v)} \overset{\underline{\text{FINITE}}}{\in} \mathbb{R} \quad (\#)$$

$(\#)$ MEANS: THE DIRECTIONAL DERIVATIVE $\dfrac{\partial f}{\partial v}(x)$ EXISTS

AND MORE $\dfrac{\partial f}{\partial v}(x) \overset{\text{Thm}}{=} L_x(v)$. !!! $\underline{\text{DONE}}$ $\underline{\text{QED}}$

PARTIAL DERIVATIVES

CONSIDER THE VECTOR SPACE $\mathbb{R}^n$

THE **CANONICAL BASIS** OF $\mathbb{R}^n$ IS THE SET:

$\{ e_1, e_2, \dots, e_n \}$ (n VECTORS IN $\mathbb{R}^n$) WHERE

FOR ANY $i = 1, 2, \dots, n$

$$\underline{e}_i = (0, 0, \dots, 0, \underset{\substack{i\text{-th} \\ \text{POS}}}{1}, 0, \dots, 0)$$

<u>EX 1</u> FOR $n = 3$, WE HAVE

$$\underline{e}_1 = (1, 0, 0), \quad \underline{e}_2 = (0, 1, 0), \quad \underline{e}_3 = (0, 0, 1).$$

<u>RMK</u> CONSIDER ANY VECTOR

$$v \overset{\text{DEF}}{=} (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$$

$$\overset{!}{=} \sum_{i=1}^{n} v_i \cdot \underline{e}_i$$

<u>IN EX 1</u>

$$v = (-1, 3, -5) \in \mathbb{R}^3$$

$$= -1 \cdot (1, 0, 0) + 3(0, 1, 0) - 5(0, 0, 1)$$

$$\overset{!}{=} -1 \cdot \underline{e}_1 + 3\underline{e}_2 - 5 \cdot \underline{e}_3 \quad !!!$$

<u>NOW</u> THE VECTORS $\underline{e}_i$ ARE DIRECTIONS,

THAT IS $\|\underline{e}_i\| = 1$!!!!!

THEN, GIVEN A FUNCT. $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$

WE CONSIDER THE DIRECTIONAL DERIVATIVES

$\longrightarrow \dfrac{\partial f}{\partial \underline{e}_i}(\underline{x})$ ARE CALLED PARTIAL DERIVATIVES !!!

TWO QUESTIONS?

1) WHY "PARTIAL"

2) WHY THE N

$\dfrac{\partial f}{\partial x_i}(\underline{x})$ IN PLACE OF $\dfrac{\partial f}{\partial \underline{e}_i}(\underline{x})$???

STOP QUESTIONS?

BYE BYE !!!