Monday, November 21, 2022 11:09 AM

HELLO!!!

I UPLOADED THE (RIGHT(?)) FILE OF LEZ 7 BIS ...

BEGIN AT 14.10

QUESTIONS?

PARTIAL DERIVATIVES

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

DEF. THE i-th ($i = 1, 2, \dots, n$) PARTIAL DERIVATIVE OF f AT $\underline{x} \in A$ IS BY DEFINITION:

THE SPECIAL DIRECTIONAL DERIVATIVE:

$$\frac{\partial f}{\partial \underline{e}_i}(\underline{x})$$, WHERE

$\underline{e}_i$ IS THE i-th ELEMENT OF THE CANONICAL BASIS $\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n\}$ OF $\mathbb{R}^n$,

THAT IS $\underline{e}_i = (0, 0, \dots, 0, \underset{i\text{-th pos}}{1}, 0, \dots, 0)$.

WHY? 1) PARTIAL?

2) WE WRITE $\frac{\partial f}{\partial x_i}(\underline{x})$ IN PLACE OF

(DEF) $\frac{\partial f}{\partial \underline{e}_i}(\underline{x})$?

LET US EXAMINE THE PREVIOUS DEF!

$$\frac{\partial f}{\partial \underline{e}_i}(\underline{x}) \overset{DEF}{=} \exists \text{ FINITE } \lim_{t\to 0} \frac{f(\underline{x}+t\underline{e}_i)-f(\underline{x})}{t} \qquad (*)$$

WE WRITE $(*)$ IN THE VARIABLE NOTATION !!!

NOTICE THAT $\underline{x} = (x_1, x_2, \dots, x_n)$

$\underline{e}_i = (0, 0, \dots, 0, \underset{\uparrow\, i\text{-th}}{1}, 0, \dots 0)$

IMPLIES THAT $t \in \mathbb{R}$

$$\underline{x} + t\underline{e}_i = (x_1, \dots, \overset{i\text{-th}}{x_i + t}, \dots, x_n)$$

SO $(*)$ BECOMES:

$$\lim_{t\to 0} \frac{f(\underline{x}+t\underline{e}_i)-f(\underline{x})}{t} =$$

$$\lim_{t\to 0} \frac{f(x_1, \dots, \overset{i\text{-th}}{x_i+t}, \dots, x_n) - f(x_1, \dots, x_i, \dots x_n)}{t} \quad !!! \ (**)$$

$(**)$ IS THE SAME AS THE ORDINARY (ONE VARIABLE) CONDITION, WHERE x_i IS JUST THE ONLY VARIABLE !!!

AND THE VARIABLES $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ ARE KEPT TO BE CONSTANT !!!

$$\frac{\partial f}{\partial x_i}(\underline{x}).$$

EX 1 $\quad n = 2 \qquad f : \mathbb{R}^2 \to \mathbb{R}, \ f(x,y) = e^{x^2 y}.$

COMPUTE THE TWO PARTIAL DERIVATIVES AT A GENERIC POINT $(x,y) \in \mathbb{R}^2$!!!

1) $\dfrac{\partial f}{\partial x}(x,y) \quad \left(\overset{DEF}{=} \dfrac{\partial f}{\partial \underline{e}_1}(x,y) \right) \qquad ???$

NOW

$$\frac{\partial f}{\partial x}(x,y) = (2xy)\cdot e^{x^2 y}$$

2) $\frac{\partial f}{\partial y}(x,y) \quad \left(\overset{DEF}{=} \frac{\partial f}{\partial e_2}(x,y) \right)$

$(x,y) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial y}(x,y) = x^2 \cdot e^{x^2 y}$$

OK ???

EX 2 $\quad f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = x^5 y + x^2 z^3 + xyz$

1) $\frac{\partial f}{\partial x}(x,y,z) = 5x^4 y + 2xz^3 + yz$

2) $\frac{\partial f}{\partial y}(x,y,z) = x^5 + xz$

3) $\frac{\partial f}{\partial z}(x,y,z) = 3x^2 z^2 + xy$

???

RECALL (FROM TUESDAY)

THM $\quad f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

IF f DIFFERENTIABLE AT $\underline{x} \in A$

THEN $\quad \forall$ DIRECTION $\|v\| = 1$

$\exists \; \frac{\partial f}{\partial v}(\underline{x}) \overset{MORE}{=} L_{\underline{x}}(v)$???

$\uparrow$ DIFFERENTIAL

QUESTION $\quad$ GIVEN $v \in \mathbb{R}^n$, HOW TO

COMPUTE THE EVALUATION

???

$L_{\underline{x}}(v)$!!!

RECALL

$v \overset{\text{DEF}}{=} (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

RECALL $\overset{!}{=} \displaystyle\sum_{i=1}^{n} v_i \cdot \underline{e}_i$!!! $(\dagger)$

FROM EQ $(\dagger)$ $\Rightarrow$

$$L_{\underline{x}}(v) = L_{\underline{x}}\left(\sum_{i=1}^{n} v_i \cdot \underline{e}_i\right) \overset{\text{LINEARITY}}{=}$$

$$= \sum_{i=1}^{n} \underbrace{v_i}_{\text{COEFF.}} \cdot L_{\underline{x}}(\underline{e}_i)$$

THIS IS THE i-th PARTIAL DERIVATION !

THM

$$= \sum_{i=1}^{n} v_i \cdot \frac{\partial f}{\partial \underline{e}_i}(\underline{x}) = \sum_{i=1}^{n} v_i \cdot \frac{\partial f}{\partial x_i}(\underline{x}) \quad !!! \quad (\#)$$

NOW, $(\#)$ SUGGESTS THE FOLLOWING DEF:

gradient of f AT $\underline{x} \in A \overset{\text{DEF}}{=}$

$$\operatorname{grad} f(\underline{x}) \overset{\text{DEF}}{=} \left(\frac{\partial f}{\partial x_1}(\underline{x}), \frac{\partial f}{\partial x_2}(\underline{x}), \dots, \frac{\partial f}{\partial x_n}(\underline{x})\right) \in \mathbb{R}^n.$$

THEN $(\#)$ BECOMES:

$$L_{\underline{x}}(v) = \langle v, \operatorname{grad} f(\underline{x}) \rangle \overset{\text{symm}}{=} \langle \operatorname{grad} f(\underline{x}), v \rangle \quad !!!$$

$\forall\, v \in \mathbb{R}^n$

IN PARTICULAR, IF v DIRECTION, $\|v\| = 1$ (THM' ! IMPLIES)

$$L_{\underline{x}}(v) = \underline{\langle \operatorname{grad} f(\underline{x}), v \rangle} \overset{THM}{=} \frac{\partial f}{\partial v}(\underline{x}) \quad !!!$$

EX $f : \mathbb{R}^2 \to \mathbb{R}$, $\underline{f(x,y) = x^3 y + x y^2}$ (DIFFERENTIABLE? LATER)

$\underline{x} = (1, -1)$, $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ DIRECTION

$$\frac{\partial f}{\partial v}(\underline{x}) \quad ???$$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 y + y^2 \Big/_{(x,y) = (1,-1)} = -3+1 = \underline{\underline{-2}} \qquad \underline{\text{IS OK?}}$$

$$\frac{\partial f}{\partial y}(x,y) = x^3 + 2xy \Big/_{(x,y) = (1,-1)} = 1 - 2 = \underline{\underline{-1}}$$

THEN $\displaystyle \frac{\partial f}{\partial v}(\underline{x}) = \frac{\partial f}{\partial \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}(1,-1) \overset{THM}{=}$

$$= \langle \operatorname{grad} f(\underline{x}), v \rangle = \left\langle (-2,-1), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \right\rangle =$$

$$= -2 \cdot \frac{1}{2} - 1 \cdot \frac{\sqrt{3}}{2}$$

QED !!!

BREAK

QUESTIONS?

BEGIN AGAIN AT 15.10 ?