Monday, November 21, 2022 11:10 AM

THM $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

IF f DIFFERENTIABLE AT $\underline{x} \in A$

THEN f IS CONTINUOUS AT $\underline{x} \in A$!!!

PROOF THESIS : f CONTINUOUS AT $\underline{x} \in A \overset{DEF}{\iff}$

$$(0 \leq)\ |f(\underline{x}+\underline{h}) - f(\underline{x})| \xrightarrow[\underline{h} \to \underline{0} \in \mathbb{R}^n]{} 0 \quad !!! \qquad (*)$$

BY ASSUMPTION : WRITE :

$$E_{\underline{x}}(\underline{h}) = f(\underline{x}+\underline{h}) - f(\underline{x}) - L_{\underline{x}}(\underline{h})$$

$$\Updownarrow$$

$$f(\underline{x}+\underline{h}) - f(\underline{x}) = E_{\underline{x}}(\underline{h}) + L_{\underline{x}}(\underline{h}) \qquad (\dagger)$$

$$\Downarrow$$

$$(\#) \qquad 0 \leq |f(\underline{x}+\underline{h}) - f(\underline{x})| \leq |E_{\underline{x}}(\underline{h})| + |L_{\underline{x}}(\underline{h})|$$

$0 \xleftarrow[\underline{h}\to 0]{}$ (?) $\xrightarrow[\underline{h}\to 0]{}$ 0

NOW, BY DIFFERENTIABILITY CONDITION $\implies$

$$\lim_{\underline{h} \to \underline{0} \in \mathbb{R}^n} \frac{E_{\underline{x}}(\underline{h})}{\|\underline{h}\|} = 0 \ !!! \implies$$

$$\implies \lim_{\underline{h} \to \underline{0} \in \mathbb{R}^n} E_{\underline{x}}(\underline{h}) = 0 \iff \lim_{\underline{h} \to \underline{0} \in \mathbb{R}^n} |E_{\underline{x}}(\underline{h})| = 0 \ !!!$$

WHAT ABOUT $|L_{\underline{x}}(\underline{v})| \xrightarrow[\underline{h} \to \underline{0} \in \mathbb{R}^n]{}$???

BUT, FROM THE PREVIOUS THM

$$|L_{\underline{x}}(\underline{v})| = |\langle \operatorname{grad} f(\underline{x}), \underline{v} \rangle| \leq$$

CAUCHY-SCHWARZ

$$\leq \|\operatorname{grad} f(\underline{x})\| \cdot \|\underline{h}\|$$

$= \| \delta \ldots | (\ldots) | \, \| \ldots \|$

cost $\quad \downarrow h \to 0 \quad 0$

$\downarrow h \to 0$

0

IN PLAIN WORDS :

$$0 \le \left| f(\underline{x}+\underline{h}) - f(\underline{x}) \right| \le \underbrace{|E_{\underline{x}}(\underline{h})|}_{\downarrow h \to 0 \;\; 0} + \underbrace{|L_{\underline{x}}(\underline{h})|}_{\downarrow h \to 0 \;\; 0} .$$

$$\Downarrow$$

$$\left| f(\underline{x}+\underline{h}) - f(\underline{x}) \right| \xrightarrow[h \to 0]{} 0 \quad ??? \quad \text{THESIS}$$

MEANS THAT $\quad \Updownarrow$

f IS CONTINUOUS AT $\underline{x} \in A$ $\quad$ (QED) OK???

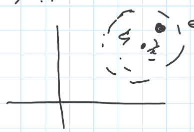
THM (TOTAL DIFFERENTIAL THEOREM)

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

HP 1 $\quad \exists \; I(\underline{x}, \delta)$ S.T. ALL THE PARTIAL DERIVATIVES EXIST ANY POINT OF THIS OPEN SPHERICAL NEIGHBOURHOOD

$I(\underline{x}, \delta)$!!!

EX IN $n = 2$



$\swarrow I(\underline{x}, \delta) \subset \mathbb{R}^2$

$\leftarrow \exists \; \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$

HP 2 $\quad$ THE PARTIAL DERIVATIVES

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$$

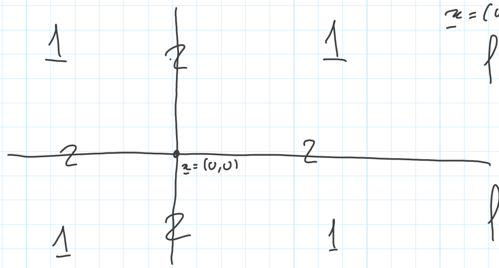
REGARDED AS FUNCTION OF THE POINTS OF $I(\underline{x}, \delta)$ (THANKS TO HP 1)

ARE CONTINUOUS AT THE CENTRAL POINT

$\underline{x} \in I(\underline{x}, \delta)$

THEN (THESIS): f DIFFERENTIABLE AT $\underline{x} \in A$!!!

COUNTEREXAMPLE $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} 2 & xy = 0 \\ 1 & xy \neq 0 \end{cases}$



$\underline{x} = (0,0)$

$f(\underline{x}) = f(0,0) = 2$!!!

BUT, WE KNOW THAT

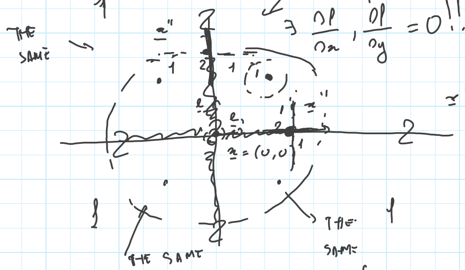
f IS NOT CONTINUOUS AT $\underline{x} = (0,0)$

$\Downarrow$

f IS NOT DIFFERENTIABLE AT $\underline{x} = (0,0)$!!!

$I(\underline{x}, \delta)$

$\exists \; \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0$!!!



$\underline{x}' = (x,0)$, $\underline{x}' \neq \underline{x}$

$\exists \; \frac{\partial f}{\partial x}(\underline{x}') = 0$ BUT !!!

$\nexists \; \frac{\partial f}{\partial y}$

$\underline{x}'' = (0,y)$, $\underline{x}'' \neq \underline{x}$

$\exists \; \frac{\partial f}{\partial \underline{e}_2}(\underline{x}'') = 0$

$= \frac{\partial f}{\partial y}(\underline{x}'')$

BUT $\nexists \; \frac{\partial f}{\partial \underline{e}_1}(\underline{x}'') \overset{DEF}{=} \frac{\partial f}{\partial x}(x)$!!!

PROP

(NOTATION: WE WILL WRITE (IN PLACE OF $L_{\underline{x}}$)

FUNCT.

$$df(\underline{x}) = L_{\underline{x}}$$

DIFFERENTIAL

LINEAR)

$f, g: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

HP f, g DIFFERENTIABLE AT $\underline{x} \in A$.

THEN

i) $f+g$ IS DIFFERENTIABLE. MORE:

$$d(f+g)(\underline{x}) \overset{\text{THM}}{=} df(\underline{x}) + dg(\underline{x})$$

ii) fg IS DIFFERENTIABLE. MORE:

$$d(fg)(\underline{x}) = \underbrace{f(\underline{x})}_{\in\mathbb{R}}\,\underbrace{dg(\underline{x})}_{\text{LINEAR FUNCT}} + \underbrace{df(\underline{x})}_{\text{LINEAR FUNC}} \cdot \overbrace{g(\underline{x})}^{\in\mathbb{R}}$$

iii) $\lambda \in \mathbb{R}$, λf IS DIFFERENTIABLE. MORE

$$d(\lambda f)(\underline{x}) = \lambda \cdot df(\underline{x}).$$

STOP

QUESTIONS?

BYEBYE GOOD WEEKEND