Monday, November 21, 2022 11:10 AM

HELLO!!!

EXAMS

YOU CAN TAKE THE EXAM FROM THE END OF THE COURSE;

EVEN DURING "HOLYDAYS",

(ONLINE, BY MAIL APPOINTMENT)

A.B.

BEGIN AT 14.10

WE RECALL:

THM $f, g : A \subseteq \mathbb{R}^n$, A OPEN, $\underline{x} \in A$.

ASSUME THAT f, g ARE DIFFERENTIABLE AT $\underline{x} \in A$.

THEN

1) $f+g$ IS DIFFERENTIABLE AT $\underline{x} \in A$. MORE:

$$d(f+g)(\underline{x}) = df(\underline{x}) + dg(\underline{x})$$

2) fg IS DIFFERENTIABLE AT $\underline{x} \in A$. MORE:

$$d(fg)(\underline{x}) = \underbrace{f(\underline{x})}_{\text{CONST} \in \mathbb{R}} \cdot \underbrace{dg(\underline{x})}_{\text{LINEAR FUNC.}} + \underbrace{df(\underline{x})}_{\text{LINEAR FUNCT.}} \cdot \underbrace{g(\underline{x})}_{\text{CONST} \in \mathbb{R}}$$

3) $\lambda \in \mathbb{R}$,

(λf) IS DIFFERENTIABLE AT $\underline{x} \in A$. MORE.

$$d(\lambda f)(\underline{x}) = \underbrace{\lambda}_{\text{SCALAR}} \cdot df(\underline{x})$$

APPLICATION IF $n = 1$, WE RECALL THAT

f IS DIFFERENTIABLE AT $x \in A$ $\overset{\text{THM}}{\Longleftrightarrow}$ f ADMITS DERIVATIVE $f'(\underline{x})$: MORE,

$$f'(\underline{x}) = L_{\underline{x}}(1) = df(\underline{x})(1). \quad (*)$$

THEN

(1) $\Rightarrow d(f+g)(x)(1) = df(x)(1) + dg(x)(1)$

$$(f+g)'(x) \overset{\text{THM}}{=} f'(x) + g'(x) \quad !!!$$

(2) $\Rightarrow$

$$d(fg)(x)(1) = f(x)\cdot dg(x)(1) + df(x)\cdot g(x)(1)$$

$$(fg)'(x) \overset{\text{THM}}{=} f(x)\,g'(x) + f'(x)\cdot g(x)$$

(3) $\Rightarrow$ $\lambda \in \mathbb{R}$

$$d(\lambda f)(x)(1) = \lambda \cdot df(x)(1)$$

$$(\lambda f)'(x) \overset{\text{THM}}{=} \lambda \cdot f'(x). \quad !!!$$

IS IT CLEAR?

HOW TO WRITE A DIFFERENTIAL INTO AN "INTRINSIC WAY" ???

WE RECALL: $f \in A \subseteq \mathbb{R}^n$, A OPEN, $x \in A$.

ASSUME THAT f DIFF AT $x \in A$.

THEN $v \in \mathbb{R}^n$,

$$L_{\underline{x}}(v) \overset{\text{NOTATION}}{=} df(\underline{x})(v) \overset{\text{THM}}{=}$$

$$\overset{\text{THM}}{=} \langle \operatorname{grad} f(\underline{x}), v \rangle \overset{\text{DEF}}{=}$$

$$\overset{\text{DEF}}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}) \cdot v_i$$

!!!...

DUAL SPACES

WE HAVE, FOR $n \in \mathbb{Z}^+$,

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) ; \quad x_i \in \mathbb{R}, \ i = 1, 2, \dots, n \}.$$

THE DUAL SET

$(\mathbb{R}^n)^*$ OF $\mathbb{R}^n$

IS

$$(\mathbb{R}^n)^* \overset{\text{DEF}}{=} \{ \varphi : \mathbb{R}^n \to \mathbb{R} ; \ \varphi \text{ LINEAR} \}$$

!!!

BUT, WE CAN DEFINE TWO OPERATIONS

ON $(\mathbb{R}^n)^*$!!!

1) $\varphi, \psi \in (\mathbb{R}^n)^*$ (φ, ψ LINEAR)

$\forall x \in \mathbb{R}^n$

$$(\varphi + \psi)(x) \overset{DEF}{=} \varphi(x) + \psi(x).$$

THEN (OBVIOUS), $\varphi + \psi$ IS LINEAR $\Leftrightarrow$

$$\varphi + \psi \in (\mathbb{R}^n)^*.$$

2) $\lambda \in \mathbb{R}$ SCALAR, $\forall x \in \mathbb{R}^n$

$(\lambda \varphi)(x) = \lambda \varphi(x)$ IS LINEAR, THAT IS

$$\lambda \varphi \in (\mathbb{R}^n)^*.$$

MAIN FACT.

$\Big((\mathbb{R}^n)^*, \underset{\text{SUM}}{+}, \underset{\text{SCALAR MULT.}}{\cdot}\Big)$ IS VECTOR SPACE,

THAT IS CALLED : THE DUAL SPACE

$(\mathbb{R}^n)^*$ OF THE SPACE $\mathbb{R}^n$.

WHAT IS THE ZERO VECTOR OF THE

$(\mathbb{R}^n)^*$???

DUAL SPACE $(\mathbb{R}^n)$!!!

ANSWER : THE ZERO VECTOR OF $(\mathbb{R}^n)^*$ IS:

$\underline{\underline{0}} : \mathbb{R}^n \to \mathbb{R}$, $\underline{0}(x) = 0 \in \mathbb{R}$ $\forall x \in \mathbb{R}^n$

LINEAR FUNCTION.

INDEED, WE HAVE: $\varphi \in (\mathbb{R}^n)^*$

$$\varphi + \underline{\underline{0}} \overset{?}{=} \varphi \overset{?}{=} \underline{\underline{0}} + \varphi$$

$\Downarrow$

$\forall x \in \mathbb{R}^n$

$$\varphi(x) + \underbrace{\underline{0}(x)}_{\overset{\shortparallel}{0}\ \text{DEF}} \overset{?}{=} \varphi(x) \overset{?}{=} \underbrace{\underline{0}(x)}_{\overset{\shortparallel}{0}\ \text{DEF}} + \varphi(x)$$

YES !!!

HOW TO DESCRIBE THE DUAL SPACES

$(\mathbb{R}^n)^*$???

IN PARTICULAR: WHAT IS THE DIMENSION

$d((\mathbb{R}^n)^*)$ OF THE DUAL SPACE $(\mathbb{R}^n)^*$???

BREAK QUESTIONS?

BEGIN AGAIN AT 15.10