Monday, November 21, 2022 11:11 AM

OK. BEGIN AGAIN

AT 15.10

QUESTIONS?

RECALL THAT

$\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n\}$ THE "CANONICAL BASIS" OF $\mathbb{R}^n$,

THAT IS, $j = 1, 2, \dots, n$

$$\underline{e}_j = (0, 0, \dots, 0, \underset{j\text{-th}}{1}, 0, \dots, 0).$$

FOR EXAMPLE: $n = 3$.

$\underline{e}_1 = (1, 0, 0)$, $\underline{e}_2 = (0, 1, 0)$, $\underline{e}_3 = (0, 0, 1)$!!!

FOR EVERY $i = 1, 2, \dots, n$, SET:

$$dx_i : \mathbb{R}^n \to \mathbb{R} \quad (\text{LINEAR})$$

SUCH THAT:

$$dx_i(\underline{e}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \overset{\text{KRONECKER SYMBOL}}{\delta_{ij}} \quad (\underline{\text{LINEAR}}) \qquad (\dagger)$$

EX IN $n = 3$

$dx_1(\underline{e}_1) = 1$, $dx_1(\underline{e}_2) = 0$, $dx_1(\underline{e}_3) = 0$

$dx_2(\underline{e}_1) = 0$, $dx_2(\underline{e}_2) = 1$, $dx_2(\underline{e}_3) = 0$!!!

$dx_3(\underline{e}_1) = 0$, $dx_3(\underline{e}_2) = 0$, $dx_3(\underline{e}_3) = 1$

SO, $\forall\, i = 1, 2, \dots, n$

$$dx_i : \mathbb{R}^n \to \mathbb{R} \text{ LINEAR} \iff dx_i \in (\mathbb{R}^n)^* \text{ !!!}$$

NOW, LET $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

$$\overset{!}{=} \sum_{j=1}^{n} v_j \cdot \underline{e}_j. \qquad (**)$$

NOW, COMPUTE

$$\underline{dx_i(v)} \overset{(**)}{=} dx_i\left(\sum_{j=1}^{n} v_j \cdot \underline{e}_j\right) \underset{=}{\overset{\text{LINEARITY}}{}}$$

$$= \sum_{j=1}^{n} v_j \cdot \underbrace{dx_i(\underline{e}_j)}_{\delta_{ij}} = v_i \qquad \text{!!!} \qquad (\dagger\dagger)$$

THEREFORE, IT IS NATURAL TO CALL THE FUNCTION $dx_i \in (\mathbb{R}^n)^*$ THE <u>COORDINATE FUNCTIONS</u> ???

<u>CRUCIAL FACT</u>

<u>THM</u> THE SET

$$\{dx_1, dx_2, \dots, dx_n\}$$

IS A <u>BASIS</u> FOR THE DUAL SPACE

$$(\mathbb{R}^n)^* \quad ???$$

IN PARTICULAR, IT FOLLOWS:

$$d((\mathbb{R}^n)^*) \overset{!!!}{=} d(\mathbb{R}^n) \quad ???$$

<u>PROOF</u> ($\{dx_1, \dots, dx_n\}$ IS A SYSTEM OF GENERATORS FOR $(\mathbb{R}^n)^*$).

SO, WE CONSIDER

$$\varphi : \mathbb{R}^n \to \mathbb{R} \text{ LINEAR} \iff \varphi \in (\mathbb{R}^n)^*.$$

LET $\underline{e}_j$ AN ELEMENT OF THE CANONICAL BASIS OF $\mathbb{R}^n$.

$$v = (v_1, \dots, v_n) \in \mathbb{R}^n \overset{!}{=} \sum_{j=1}^{n} v_j \cdot \underline{e}_j$$

CONSIDER

$$(*) \begin{cases} \varphi(v) = \varphi\left(\sum_{j=1}^{n} v_j \cdot \underline{e}_j\right) \overset{LIN !!!}{=} \\ = \sum_{j=1}^{n} \underbrace{v_j}_{dx_j(v)} \cdot \varphi(\underline{e}_j) = \sum_{j=1}^{n} \varphi(\underline{e}_j) \cdot dx_j(v) \end{cases} \quad \forall v \in \mathbb{R}^n$$

???

$\Downarrow$ ← LINEAR COMB OF THE dx_j !!!

n ... ???

(**) $$\varphi = \sum_{j=1} \underset{\substack{\uparrow \\ \text{COEFF} \\ \text{IN } \mathbb{R}}}{\varphi(\underline{e}_j)} \cdot dx_j \quad \text{IN } (\mathbb{R}^n)^* \; !!!$$

QED OK !!!

(2) (LINEAR INDEPENDENCE). WE HAVE TO PROVE

(*) THAT $\{dx_1, dx_2, \dots, dx_n\}$ IS

LINEAR INDEPENDENT IN $(\mathbb{R}^n)^*$!!!

(*) MEANS:

(***) $$\sum_{i=1}^{n} \underset{\substack{\uparrow \\ \mathbb{R}}}{c_i}\, dx_i = \overset{\substack{\text{ZERO VECTOR} \\ \text{IN } (\mathbb{R}^n)^* \\ \downarrow}}{\underline{\underline{0}}} \Longrightarrow c_1 = c_2 = \dots = c_n = 0 \; !!!$$

(***) IS OUR THESIS !!!

NOW, SPECIALIZE TO EVALUATION ON $\underline{e}_1$!!!

$$\sum_{i=1}^{n} c_i \cdot dx_i(\underline{e}_1) = \underline{\underline{0}}(\underline{e}_1) = \underline{0} \in \mathbb{R}$$

$$\Big\downarrow \delta_{ij}$$

$$\underline{c_1 \cdot dx_1(\underline{e}_1)} = c_1$$

$$\Longrightarrow c_1 = 0 \; !!!$$

NOW, SPECIALIZE TO THE EVALUATION ON $\underline{e}_2$.

$$\sum_{i=1}^{n} c_i\, dx_i(\underline{e}_2) = \underline{\underline{0}}(\underline{e}_2) = 0 \in \mathbb{R}$$

$$\Big\downarrow \delta_{i2}$$

$$\Longrightarrow c_2 = 0 \; !!!$$

$$c_2 \cdot dx_2(\underline{e}_2) = c_2$$
$$\overset{\shortparallel}{1}$$

REPEAT THE PROCEDURE UP TO $\underline{e}_n$.

$$\sum_{i=1}^{n} c_i dx_i(\underline{e}_n) = \underline{0}(\underline{e}_n) = 0 \in \mathbb{R}$$
$$\downarrow$$
$$c_n \cdot dx_n(\underline{e}_n) = c_n$$
$$\Rightarrow c_n = 0 \quad !!!$$

THEN, $c_1 = c_2 = \dots = c_n = 0 \implies$
$\{dx_1, dx_2, \dots dx_n\}$ IS LINEARLY INDEPENDENT IN $(\mathbb{R}^n)^*$. QED OK.

RMK WE HAVE (RECALL) THAT

$$\forall v \in \mathbb{R}^n, \quad dx_i(v) = v_i \qquad (\dagger)$$

SO, WRITTEN IN THE "VARIABLE NOTATION",
$dx_i : \mathbb{R}^n \to \mathbb{R}$, $dx_i(x_1, x_2, \dots, x_n)$, WHAT ARE THE dx_i ???

ANSWER IS:

$$dx_i(x_1, x_2, \dots, x_n) = x_i$$

THE HOMOGENEOUS 1 DEGREE POLYNOMIAL IN WHICH THE ONLY VARIABLE x_i APPEARS!

EX $n = 3$

$dx_1(x_1, x_2, x_3) = x_1$, $dx_2(x_1, x_2) = x_2$, $dx_3(x_1, x_2, x_3) = x_3$???

CAN YOU SEE/HEAR ME !??!

PLEASE, GIVE A FEEDBACK

SO, $\varphi \in (\mathbb{R}^n)^*$, φ LINEAR $\Longrightarrow$

$$\varphi(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \varphi(e_i) \cdot x_i$$

$\varphi(e_i)$: COEFF IN $\mathbb{R}$

$\Downarrow$

$\varphi(x_1, x_2, \dots, x_n)$ IS A HOMOGENEOUS POLYNOMIAL OF DEGREE 1

(WITH COSTANT TERM EQUAL TO 0).

IN $n=1$, WE ALREADY KNOW:

$$\varphi : \mathbb{R} \to \mathbb{R} \text{ LINEAR} \iff \varphi(x) = K \cdot x$$!!! OK.

IN $n=3$, WE HAVE

1) $\varphi(x_1, x_2, x_3) = 3x_1 - 2x_2 + 17x_3$ LINEAR

2) $\varphi(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3$ NOT LINEAR

3) $\varphi(x_1, x_2, x_3) = 3x_1 - 2x_2 + 17x_3 + \underset{\substack{\neq \\ 0}}{1}$ NOT LINEAR !!!

STOP QUESTIONS ?

BYE BYE