Monday, November 21, 2022 11:12 AM

HELLO!!!

BEGIN AT 11.10

WE PROVED: $\varphi : \mathbb{R}^n \to \mathbb{R}$ LINEAR $\iff \varphi \in (\mathbb{R}^n)^*$

THEN:

SCALAR COEFF

$$\varphi \overset{!}{=} \sum_{i=1}^{n} \underset{\in \mathbb{R}}{\varphi(\underline{e}_i)} \cdot dx_i \qquad (*)$$

SINCE $dx_i(x_1, \dots, x_n) = x_i$

$$(*) \iff \varphi(x_1, \dots, x_n) = \sum_{i=1}^{n} \underset{\in \mathbb{R}}{\overset{=K_i \in \mathbb{R}}{\varphi(\underline{e}_i)}}\, x_i$$

THEN, $\varphi(x_1, \dots, x_n)$ LINEAR $\overset{\text{THM}}{\iff}$

$$\varphi(x_1, \dots, x_n) = \sum_{i=1}^{n} K_i x_i$$ LINEAR

$\Updownarrow$

φ IS A HOMOGENEOUS POLYNOMIAL OF DEGREE 1 ($\Rightarrow$ THE COSTANT TERM IS ZERO).

EX 1 $n=3$

i) $\varphi(x_1, x_2, x_3) = 3x_1 - 2x_2 + 14x_3$ LINEAR

ii) $f(x_1, x_2, x_3) = x_1 x_2^2 + x_2 x_3$ NOT LINEAR

iii) $g(x_1, x_2, x_3) = 3x_1 - 2x_2 + 14x_3 + 1$ NOT LINEAR

APPL TO DIFFERENTIAL CALCULUS

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

ASSUME THAT f DIFFERENTIABLE AT $\underline{x} \in A$.

THE DIFFERENTIAL $df(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$ LINEAR
(EXISTS)

$\Updownarrow$

$df(\underline{x}) \in (\mathbb{R}^n)^*$.

COND. (*) IMPLIES THAT

$$df(\underline{x}) \overset{!}{=} \sum_{i=1}^{n} df(\underline{x})(\underline{e}_i)\, dx_i$$

$\Downarrow$ THM

$$\overset{!}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial \underline{e}_i}(\underline{x})\, dx_i$$

$\downarrow$ PARTIAL DERIVATIVES

$$\overset{!}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x})\, dx_i$$

IN SYNTHESIS:

$$(**) \quad df(\underline{x}) \overset{!}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}) \cdot dx_i$$

BUT $dx_i(x_1, \dots, x_n) = x_i$

$$(***) \quad \overset{!}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}) \cdot x_i$$

$\swarrow$ HOMOGENEOUS POLYNOMIAL OF DEGREE 1

RMK $\forall v \in \mathbb{R}^n$ (**) IMPLIES

$$\underline{df(\underline{x})(v)} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x})\, \underline{dx_i(v)}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}) \cdot v_i$$

$$= \langle \underline{\operatorname{grad} f(\underline{x})}, v \rangle$$

(AS WE ALREADY KNOW!)

IS IT CLEAR ???...

THE MIXED (ITERATED) DERIVATIVES OF ORDER 2.

LET $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

SUPPOSE THAT $\quad i, j = 1, 2, \dots, n, \; i \neq j$

$$\exists \; \frac{\partial f}{\partial x_i}(x_1, \dots x_n) \qquad \forall (x_1, \dots, x_n) \in A$$

$\Downarrow$

$\exists \; \frac{\partial}{\,}\Big(\frac{\partial f}{\,}\big(\quad \big)\Big)$

$\swarrow$ MIXED DERIVATIVE OF ORDER 2

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(x_1,\dots,x_n)\right) \leftarrow$$

$$\| $$

$$\rightarrow \frac{\partial^{(2)} f}{\partial x_j \partial x_i}(x_1,\dots,x_n) \qquad (+)$$

(+) CAN ASK : IS THERE RELATION

BETWEEN

$$\frac{\partial^{(2)} f}{\partial x_j \partial x_i} \quad \text{and} \quad \frac{\partial^{(2)} f}{\partial x_i \partial x_j} \qquad ??? \quad \underline{\text{NO}}\ !!!$$

<u>COUNTEREXAMPLE</u> $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 y + |y|$

NOW $\frac{\partial f}{\partial x}(x,y) = 2xy$

$$\Rightarrow \exists \frac{\partial^{(2)} f}{\partial y \partial x}(x,y) \overset{\text{DEF}}{=} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$= \frac{\partial f}{\partial y}(2xy) = 2x \quad !!!$$

BUT WHAT ABOUT

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(x^2 y + |y|)$$

THIS DOES NOT EXIST FOR $\underline{y=0}$!!!

THEN

$$\nexists \frac{\partial^{(2)} f}{\partial x \partial y}(x,y) \quad \text{IF} \quad y=0 !!!$$

THM (<u>SCHWARTZ THM</u>)

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

HP1 $\quad \exists \dfrac{\partial^{(2)} f}{\partial x_j \partial x_i}$, $\exists \dfrac{\partial^{(2)} f}{\partial x_i \partial x_j}$ EXIST ON

AN OPEN SPHERICAL NEIGHBORHOOD $I(\underline{x}, \delta)$ OF $\underline{x} \in A$.

HP2 $\quad$ REGARDED AS FUNCTIONS ON $I(\underline{x}, \delta)$

$\dfrac{\partial^{(2)} f}{\partial x_j \partial x_i}$ and $\dfrac{\partial^{(2)} f}{\partial x_i \partial x_j}$ ARE CONTINUOUS AT $\underline{x} \in A$.

THEN

TH $\quad \dfrac{\partial^{(2)} f}{\partial x_j \partial x_i}(\underline{x}) = \dfrac{\partial^{(2)} f}{\partial x_i \partial x_j}(\underline{x})$!!!...

EX $\quad n = 2$, $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 y + x^2 y^2$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 y + 2x y^2$$

$$\Rightarrow \frac{\partial^{(2)} f}{\partial y \partial x}(x,y) \overset{\text{DEF}}{=} \frac{\partial}{\partial y}\left(3x^2 y + 2x y^2\right) = 3x^2 + 4xy$$

|| THE SAME!!!

$$\frac{\partial f}{\partial y}(x,y) = x^3 + 2x^2 y$$

$$\Rightarrow \frac{\partial^{(2)} f}{\partial x \partial y}(x,y) \overset{\text{DEF}}{=} \frac{\partial}{\partial x}\left(x^3 + 2x^2 y\right) = 3x^2 + 4xy$$

IS IT CLEAR?!!...

BREAK QUESTIONS!

BEGIN AGAIN AT 12.05