ESERCIZI SU INTEGRALI TRIPLI

"Calcolare $\iiint_A f(x,y,z)\,dx\,dy\,dz$",
essendo $A \subseteq \mathbb{R}^3$ e $f(x,y,z)$ come di volta in volta descritte.

1) $A = \{(x,y,z) \in \mathbb{R}^3;\ |x| + |y| \leq 2,\ \ x+y \leq x+y+z \leq 4\}$
$f(x,y,z) = x$

2) $A = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 \leq 2;\ 0 \leq z \leq \sqrt{x^2+y^2}\}$
$f(x,y,z) = x^2+y^2$

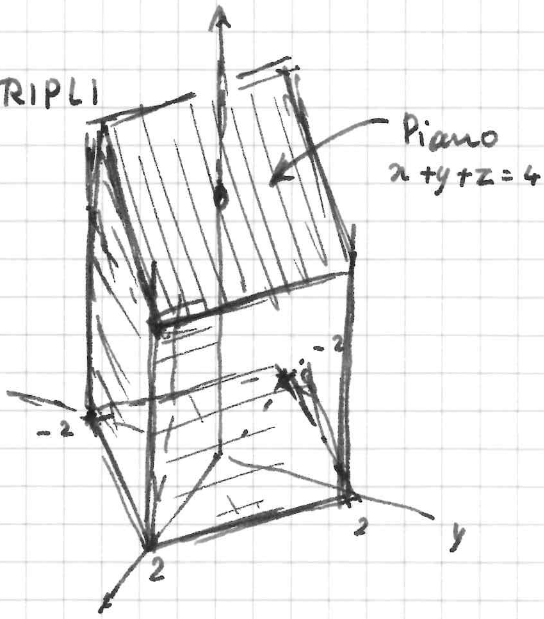
3) $A = \{(x,y,z) \in \mathbb{R}^3;\ 6x+3y+2z \leq 6;\ x \geq 0, y \geq 0, z \geq 0\}$
$f(x,y,z) = y$

4) $A = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 \leq 50;\ x^2+y^2-z^2 \leq 32\}$
$f(x,y,z) = 1$

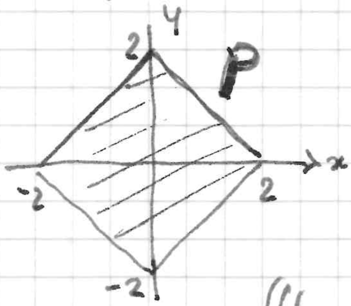
5) $A = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2 \leq z \leq 2x\}$
$f(x,y,z) = 1$

6) $A = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2-z \geq 0;\ x^2+y^2+z^2 \leq 2,\ x \geq 0, y \geq 0\}$
$f(x,y,z) = z$

SVOLGIMENTI DEGLI ESERCIZI SU INTEGRALI TRIPLI



1) La condizione $x+y \leq x+y+z \leq 4$ equivale a $0 \leq z \leq 4-x-y$.
La proiezione di A sul piano xy è $P=\{(x,y);|x|+|y| \leq 2\}$



e se $(x,y) \in P$, allora

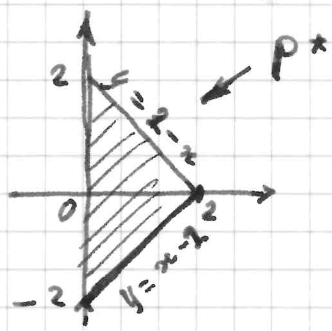
$A_{(x,y)} = \{z \in \mathbb{R};\ 0 \leq z \leq 4-x-y\}$. Allora

$$\iiint_A x\,dx\,dy\,dz = \iint_P \left(\int_0^{4-x-y} x\,dz \right) dx\,dy$$

$$= \iint_P \left(4x - x^2 - xy\right) dx\,dy = -\iint_P x^2\,dx\,dy$$

perché "$4x$" e "$-xy$" sono funzioni x-dispari e P è simmetrico rispetto all'asse y, quindi il loro integrale su P è uguale a 0.
Invece "$-x^2$" è x-pari, perciò

$$\iint_P -x^2\,dx\,dy = 2 \iint_{P^*} -x^2\,dx\,dy$$



con P^* come in figura

$$= 2 \int_0^2 \left(\int_{x-2}^{2-x} -x^2\,dy \right) dx = 4 \int_0^2 -x^2(2-x)\,dx$$

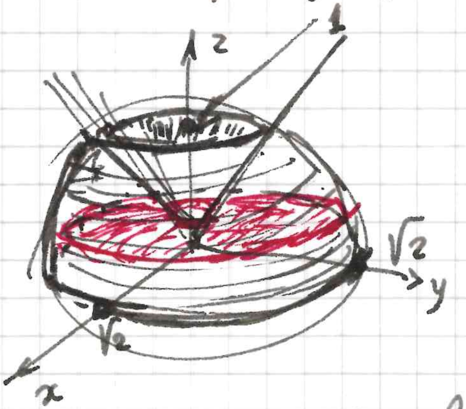
$$= 4 \int_0^2 \left(x^3 - 2x^2\right) dx = 4 \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 \right]_{x=0}^{x=2}$$

$$= 4\left(4 - \frac{16}{3}\right) = \frac{-16}{3}$$

2) $A = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 \le 2;\ 0 \le z \le \sqrt{x^2+y^2}\}$

$f(x,y,z) = x^2+y^2$

Il dominio è una semisfera con centro $(0,0,0)$ e raggio $\sqrt{2}$, con una cavità conica.



Serve conoscere a quale quota z la superficie del cono interseca quella della sfera.

Da $\begin{cases} x^2+y^2+z^2 = 2 \\ z = \sqrt{x^2+y^2} \end{cases}$ ricaviamo $x^2+y^2 = z^2$,

quindi $2z^2 = 2$, $z^2 = 1$

e infine $z=1$, poiché è stabilito che sia $z \ge 0$.

Perciò la proiezione di A sull'asse z è l'intervallo $[0,1]$. Possiamo applicare il cambiamento di variabili in coordinate cilindriche: $\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \\ z = t \end{cases}$

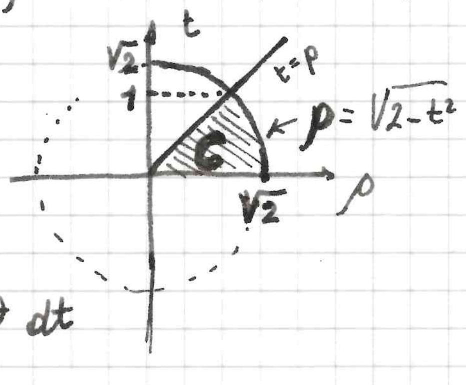
Allora $B = \{\rho \ge 0,\ \theta \in [0,2\pi],\ t \ge 0,\ (\rho\cos\theta, \rho\,\mathrm{sen}\,\theta, t) \in A\}$

è l'insieme:

$B = \{(\rho,\theta,t);\ t \in [0,1];\ t \le \rho \le \sqrt{2-t^2};\ \theta \in [0,2\pi]\}$

$= \{(\rho,\theta,t);\ \theta \in [0,2\pi],\ (\rho,t) \in C\}$

con C come in figura



Allora:

$$\iiint_A (x^2+y^2)\,dx\,dy\,dz = \iiint_B \rho^2 \cdot \underbrace{\rho}_{|\det J|}\, d\rho\, d\theta\, dt$$

$|\det J|$ (jacobiano della trasformazione)

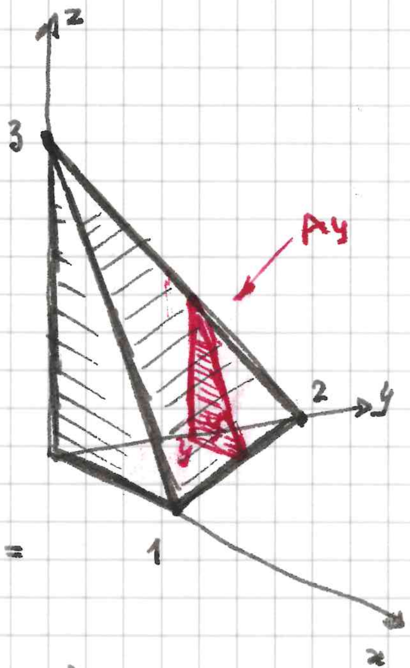
$$= \int_0^{2\pi} \left(\iint_C \rho^3\, d\rho\, dt \right) d\theta = \int_0^{2\pi} \left(\int_0^1 \left(\int_t^{\sqrt{2-t^2}} \rho^3\, d\rho \right) dt \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_0^1 \left[\rho^4\right]_{\rho=t}^{\rho=\sqrt{2-t^2}} dt \right) d\theta = \int_0^{2\pi} \left(\int_0^1 (4-4t^2)\, dt \right) d\theta$$

$$= \int_0^{2\pi} \left[4t - \frac{4}{3}t^3\right]_{t=0}^{t=1} d\theta = \frac{8}{3}\int_0^{2\pi} 1\, d\theta = \frac{16}{3}\pi.$$

3) $A = \{(x,y,z) \in \mathbb{R}^3;\ x \geq 0,\ y \geq 0,\ z \geq 0,\ 6x + 3y + 2z \leq 6\}$

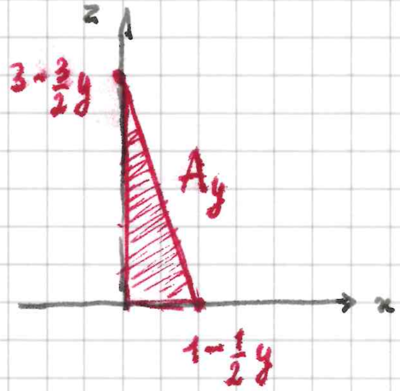
$f(x,y,z) = y$



1° modo: "a strati" paralleli ~~all'asse y~~ al piano xz

Per ogni $y \in P = \{y \in \mathbb{R};\ 0 \leq y \leq 2\}$ è

$A_y = \{(x,z) \in \mathbb{R}^2;\ x \geq 0,\ z \geq 0,\ 6x + 2z \leq 6 - 3y\}$



e allora

$$\iiint_A y\,dx\,dy\,dz =$$

$$= \int_0^2 \left(\iint_{A_y} y\,dx\,dz \right) dy$$

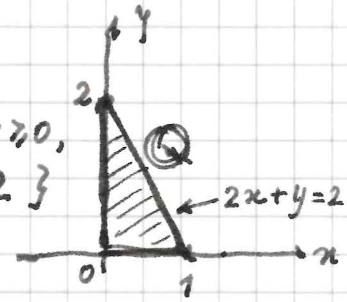
$$= \int_0^2 y \cdot \text{Area di } A_y\ dy$$

$$= \int_0^2 y \cdot \frac{3}{2}\left(1 - \frac{1}{2}y\right)^2 dy = \frac{3}{2}\int_0^2 \left(y - y^2 + \frac{1}{4}y^3\right) dy$$

$$= \frac{3}{2}\left[\frac{1}{2}y^2 - \frac{1}{3}y^3 + \frac{1}{16}y^4\right]_{y=0}^{y=2} = \frac{3}{2}\cdot\left(2 - \frac{8}{3} + 1\right) = \frac{1}{2}$$

2° modo "a fili" paralleli all'asse z.

Per ogni $(x,y) \in Q = P_{1,2}(A) = \{(x,y) \in \mathbb{R}^2,\ x \geq 0,\ y \geq 0,\ 2x + y \leq 2\}$



si ha $A_{(x,y)} = \{z \in \mathbb{R};\ 0 \leq z \leq 3 - 3x - \frac{3}{2}y\}$

e allora

$$\iiint_A y\,dx\,dy\,dz = \iint_Q \left(\int_0^{3-3x-\frac{3}{2}y} y\,dz \right) dx\,dy = \iint_Q \left(3y - 3xy - \frac{3}{2}y^2\right) dx\,dy$$

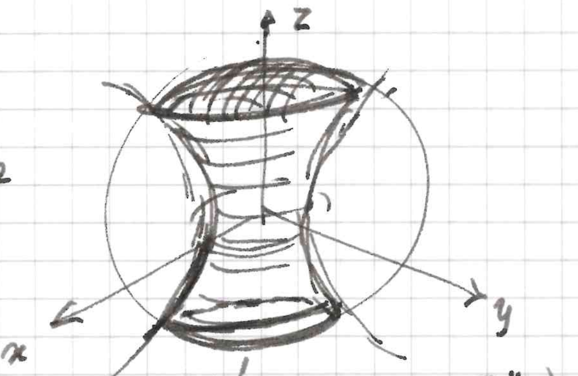
$$= \int_0^2 \left(\int_0^{1-\frac{1}{2}y} \left(3y - 3xy - \frac{3}{2}y^2\right) dx \right) dy = \int_0^2 \left[3xy - \frac{3}{2}x^2 y - \frac{3}{2}xy^2\right]_{x=0}^{x=1-\frac{1}{2}y}$$

$$= \int_0^2 \left(3\left(1 - \frac{1}{2}y\right)y - \frac{3}{2}\left(1 - \frac{1}{2}y\right)^2 y - \frac{3}{2}\left(1 - \frac{1}{2}y\right)y^2\right) dy = (\ldots)$$

(il 1° modo è senz'altro più veloce)

4) $A = \{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2+z^2 \le 50\ ;\ x^2+y^2-z^2 \le 32\}$

$f(x,y,z) = 1$

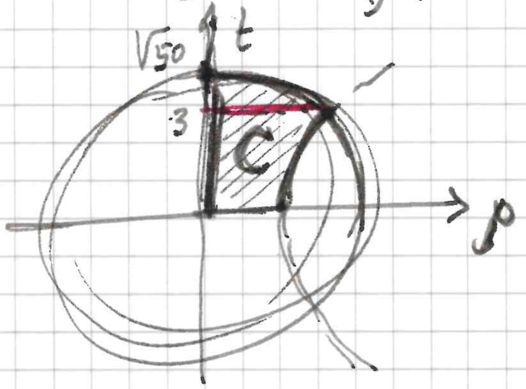


Usiamo il cambiamento di variabili in coordinate cilindriche

$$\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = t \end{cases}$$

Abbiamo allora (con il consueto significato per "B")

$$B = \{(\rho,\theta,t)\ ;\ \theta\in[0,2\pi];\ \rho^2+t^2\le 50\ ;\ \rho^2-t^2\le 32,\ \rho\ge 0\}$$
$$= \{(\rho,\theta,t)\ ;\ \theta\in[0,2\pi],\ (\rho,t)\in C\} \text{ con}$$
$$C = \{(\rho,t);\ \rho\ge 0,\ \rho^2+t^2\le 50,\ \rho^2-t^2\le 32\}$$



e allora

$$\iiint_A 1\, dx\,dy\,dz = \iiint_B \rho\, d\rho\, d\theta\, dt =$$

$$= \int_0^{2\pi} \left(\boxed{\iint_C \rho\, d\rho\, dt} \right) d\theta$$

* [non dipende da θ]

$$= 2\pi \cdot \boxed{\iint_C \rho\, d\rho\, dt}$$

$\boxed{*}$ va calcolato in due parti: una con $0\le t\le 3$, l'altra con $3\le t\le\sqrt{50}$

I) $$\int_0^3 \left(\int_0^{\sqrt{32+t^2}} \rho\, d\rho \right) dt = \int_0^3 \left(16+\frac{1}{2}t^2\right) dt$$
$$= \left[16t+\frac{1}{6}t^3\right]_{t=0}^{t=3} = 48+\frac{9}{2} = \frac{105}{2}$$

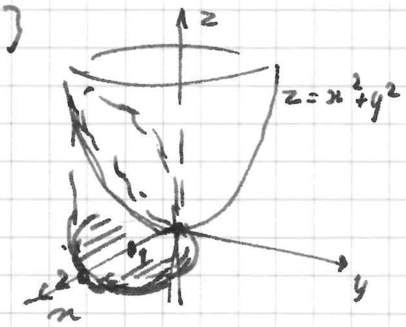
II) $$\int_3^{\sqrt{50}} \left(\int_0^{\sqrt{50-t^2}} \rho\, d\rho \right) dt = \int_3^{\sqrt{50}} \left(25-\frac{1}{2}t^2\right) dt = \left[25t-\frac{1}{6}t^3\right]_{t=3}^{t=\sqrt{50}} = \frac{50}{3}\sqrt{50}-\frac{141}{2}$$

$$(I)+(II) = \frac{50}{3}\sqrt{50}-18 = \frac{250}{3}\sqrt{2}-18$$

cosicché il volume di A è: $2\pi\cdot\left(\frac{250}{3}\sqrt{2}-18\right)$

5) $A = \{(x,y,z) \in \mathbb{R}^3 ;\ x^2+y^2 \le z \le 2x\}$

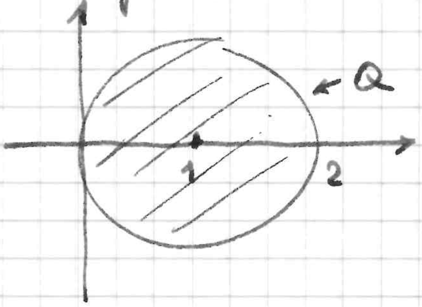
$f(x,y,z) = 1$



A è la regione di spazio compresa tra il paraboloide $z=x^2+y^2$ e il piano $z=2x$.

Per $(x,y) \in \mathbb{R}^2$ è $A_{(x,y)} = \{z \in \mathbb{R},\ x^2+y^2 \le z \le 2x\}$.

Questo insieme è $\neq \emptyset$ se $x^2+y^2 \le 2x$, ossia $x^2+y^2-2x \le 0$. La proiezione di A sul piano xy è pertanto $P_{1,2}(A) = Q = \{(x,y) \in \mathbb{R}^2 ;\ x^2+y^2-2x \le 0\}$



cioè il cerchio con centro $(1,0)$ e raggio 1. Abbiamo allora

$$\iiint_A 1\, dx\, dy\, dz = \iint_Q \left(\int_{x^2+y^2}^{2x} 1\, dz \right) dx\, dy$$

$$= \iint_Q -(x^2+y^2-2x)\, dx\, dy .$$

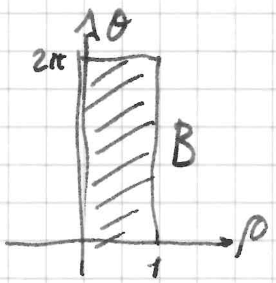
Usiamo coordinate polari centrate in $(1,0)$: $\begin{cases} x = 1+\rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$

cosicché $(x-1)^2+y^2 = \rho^2$, ossia $x^2+y^2-2x = \rho^2-1$

Allora:

$$\iint_Q -(x^2+y^2-2x)\, dx\, dy = \iint_B -(\rho^2-1)\cdot\rho\, d\rho\, d\theta \quad (*)$$

con $B = \{(\rho,\theta);\ 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi\}$

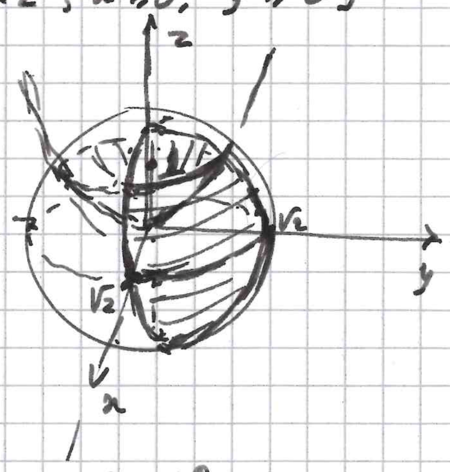


$$(*) = \int_0^1 \left(\int_0^{2\pi} (\rho-\rho^3)\, d\theta \right) d\rho$$

$$= 2\pi \int_0^1 (\rho-\rho^3)\, d\rho = 2\pi \left[\tfrac{1}{2}\rho^2 - \tfrac{1}{4}\rho^4 \right]_{\rho=0}^{\rho=1} = \frac{\pi}{2}$$

6) $A = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2-z \geq 0,\ x^2+y^2+z^2 \leq 2,\ x \geq 0,\ y \geq 0\}$

$f(x,y,z) = z$



A è delimitato dalla sfera con centro (0,0,0) e raggio $\sqrt{2}$, e dal paraboloide $z = x^2+y^2$, al quale A sta sotto. Inoltre, A si trova nel quadrante in cui $x \geq 0$, $y \geq 0$

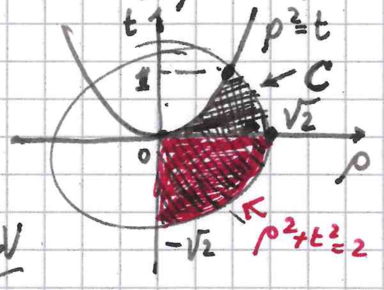
Usiamo le coordinate cilindriche $\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = t \end{cases}$

L'insieme "B" in cui debbono variare (ρ, θ, t) è:

$B = \{(\rho, \theta, t);\ \theta \in [0, \frac{\pi}{2}];\ t \leq \rho^2;\ \rho^2+t^2 \leq 2;\ \rho \geq 0\}$

$= \{(\rho, \theta, t);\ \theta \in [0, \frac{\pi}{2}];\ (\rho, t) \in C\}$

con $C = \{(\rho, t);\ \rho \geq 0,\ \rho^2 \geq t,\ \rho^2+t^2 \leq 2\}$



Allora:

$$\iiint_A z\,dx\,dy\,dz = \iiint_B \rho \cdot t\,d\rho\,d\theta\,dt =$$

$$= \int_0^{\frac{\pi}{2}} \left(\boxed{\underbrace{\iint_C \rho \cdot t\,d\rho\,dt}_{*}} \right) d\theta$$

Il calcolo di $*$ va svolto in due parti: "rosso" e "nero"

$$(\mathrm{I})\ \int_{-\sqrt{2}}^{0} \left(\int_0^{\sqrt{2-t^2}} \rho\, t\,d\rho \right) dt = \int_{-\sqrt{2}}^{0} \frac{1}{2}\left[\rho^2 t\right]_{t=0}^{t=\sqrt{2-t^2}} = \int_{-\sqrt{2}}^{0} \left(t - \frac{1}{2}t^3\right) dt$$

$$= \left[\frac{1}{2}t^2 - \frac{1}{8}t^4\right]_{t=-\sqrt{2}}^{t=0} = -\frac{1}{2}$$

$$(\mathrm{II})\ \int_0^1 \left(\int_{\sqrt{t}}^{\sqrt{2-t^2}} \rho\, t\,d\rho \right) dt = \int_0^1 \frac{1}{2}\left[\rho^2 t\right]_{\rho=\sqrt{t}}^{\rho=\sqrt{2-t^2}} dt$$

$$= \int_0^1 \left(t - \frac{1}{2}t^3 - \frac{1}{2}t^2\right) dt = \left[\frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{1}{6}t^3\right]_{t=0}^{t=1} = \frac{5}{24}$$

cosicché $(\mathrm{I})+(\mathrm{II}) = -\frac{1}{2} + \frac{5}{24} = -\frac{7}{24}$, e l'integrale originale vale $\frac{\pi}{2} \cdot \left(-\frac{7}{24}\right) = -\frac{7}{12}\pi$