Monday, November 21, 2022 11:18 AM

BEGIN AT 12.07 QUESTIONS?

THM (IMPLICIT FUNCTS THEOREM, GENERALIZE FORM OF THE DINI'S THM)

LET $F : A \subseteq \mathbb{R}^n \to \mathbb{R}^{n-r}$, A OPEN SET,

ASSUME $F \in C^{(1)}_A$.

LET $Z_F \overset{DEF}{=} \{ (x_1,\dots,x_r, y_1,\dots,y_{n-r}) ; F(x_1,\dots x_r, y_1,\dots,y_{n-r}) = \underline{0} \in \mathbb{R}^{n-r} \}$

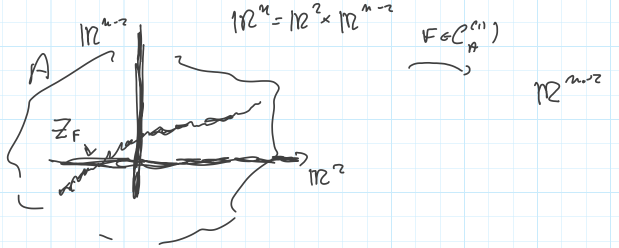
'LOCUS OF ZEROES' WE WRITE IN THIS WAY : SINCE

$\mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r}$ AND THEN A POIN IN $\mathbb{R}^n$

CAN BE WRITTEN IN THE FORM:

$$(\underbrace{x_1,\dots,x_r}_{x} ; \underbrace{y_1,\dots,y_{n-r}}_{y}) \in \mathbb{R}^n.$$

WE CAN PICTURE



SUPPOSE $(\underline{a},\underline{b}) = (\alpha_1,\dots,\alpha_r ; \beta_1,\dots,\beta_{n-r}) \in Z_F$ THAT IS

$$F(\underline{a},\underline{b}) = F(\alpha_1,\dots,\alpha_r ; \beta_1,\dots,\beta_{n-r}) = \underline{0} \in \mathbb{R}^{n-r}$$

NOW REMK

VECTOR EQS

$$(*) \quad Z_F = \{ (x_1,\dots,x_r, y_1,\dots,y_{n-r}) \in \mathbb{R}^n ; F(x_1,\dots,x_r, y_1,\dots,y_{n-r}) = \underline{0} \in \mathbb{R}^{n-r} \}$$

$\Updownarrow$

(**) A SYSTEM OF $n-r$ EQUATIONS INTO n INDETERMINATES $\to$

$$\begin{cases} F_1(x_1,\dots,x_r ; y_1,\dots,y_{n-r}) = 0 \in \mathbb{R} \\ \dots \\ \dots \\ F_{n-r}(x_1,\dots,x_r ; y_1,\dots,y_{n-r}) = 0 \in \mathbb{R} \end{cases}$$

Hp

$$J_{F(\underline{a},\underline{b})} = \left(\frac{\partial F_1}{\partial x_1}(\underline{a},\underline{b}) \ \cdots \ \frac{\partial F_1}{\partial x_r}(\underline{a},\underline{b}) \ \middle| \ \frac{\partial F_1}{\partial y_1}(\underline{a},\underline{b}) \ \cdots \ \frac{\partial F_1}{\partial y_{n-r}}(\underline{a},\underline{b}) \right.$$

$F(a,\beta)$

$(n-2)\times n$ order

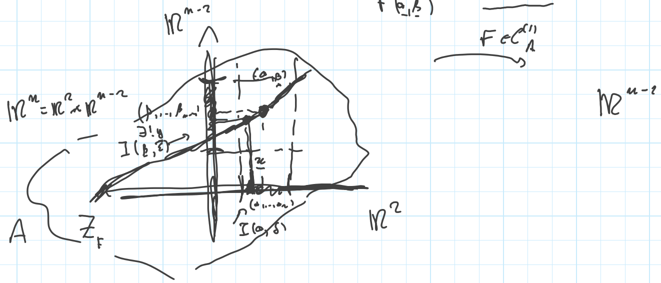
$$\left(\frac{\partial F_{n-2}}{\partial x_1}(a,\beta) \;\cdots\; \frac{\partial F_{n-2}}{\partial x_2}(a,\beta) \;\; \boxed{\frac{\partial F_{n-2}}{\partial y_1}(a,\beta) \;\cdots\; \frac{\partial F_{n-2}}{\partial y_{n-2}}(a,\beta)} \right)$$

THE LAST $n-2$ COLUMNS IS A SQUARE SUBMATRIX

HP!!!
$$\det \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(a,\beta) & \cdots & \frac{\partial F_1}{\partial y_{n-2}}(a,\beta) \\ \text{---} & & \text{---} \\ \frac{\partial F_{n-2}}{\partial y_1}(a,\beta) & \cdots & \frac{\partial F_{n-2}}{\partial y_{n-2}}(a,\beta) \end{pmatrix} \neq 0 !!!$$

THIS IMPLIES $\mathrm{rk}\, J_{F(a,\beta)} = n-2$!!!

$F \in C^1_A \longrightarrow \mathbb{R}^{n-2}$



TH 1 $\exists\, I(a,\delta) \subseteq \mathbb{R}^2 \quad \exists\, I(\beta,\varepsilon) \subseteq \mathbb{R}^{n-2}$

S.T. $\forall\, (x_1,\dots,x_2) \in I(a,\delta)$

$\exists!$ (UNIQUE) $(y_1,\dots,y_{n-2}) \in I(\beta,\varepsilon)$

S.T.

$\longrightarrow F(x_1,\dots,x_2,y_1,\dots,y_{n-2}) = \underline{0} \in \mathbb{R}^{n-2}$ THAT IS, IN TURN

$\forall\, \underline{x} = (x_1,\dots,x_2) \in I(a,\delta) \quad \exists!\, \underline{y} = (y_1,\dots,y_{n-2}) \in I(\beta,\varepsilon)$

SUCH THAT $(\underline{x},\underline{y}) \in Z_F$!!! !!!

CAN BE READ AS FOLLOWS: THE LOCUS OF ZEROES Z_F

DEFINES A FUNCTION:

$$\varphi : I(a,\delta) \longrightarrow I(\beta,\varepsilon)$$

$$\varphi : (x_1,\dots,x_2) \in I(a,\delta) \longrightarrow (y_1,\dots,y_{n-2}) \in I(\beta,\varepsilon)$$

THAT IS IN TURN

|| WE CAN WRITE :

$$\begin{cases} y_1 = \varphi_1(x_1,\dots,x_n) \\ \dots \\ y_{n-r} = \varphi_{n-r}(x_1,\dots,x_n) \end{cases}$$!!!

BUT FURTHERMORE

THR THE FUCTS $\varphi_1,\dots,\varphi_{n-r} \in C^{(1)}$!!!!!

EX LET $F:\mathbb{R}^3 \to \mathbb{R}^2$, $F \equiv (F_1, F_2)$

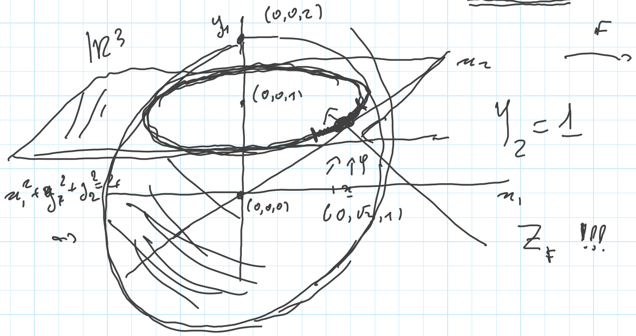
WRITE

$(x_1, y_1, y_2) \in \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1$

$$\begin{cases} F_1(x_1,y_1,y_2) = x_1^2 + y_1^2 + y_2^2 - 4 \\ F_2(x_1,y_1,y_2) = y_2 - 1 \end{cases}$$ OK

WHAT IS

$$Z_F \overset{?}{=} \{(x_1,y_1,y_2)\in\mathbb{R}^3 \; ; \; x_1^2+y_1^2+y_2^2=4 \; , \; y_2=1\}$$



$y_2 = 1$

$\mathbb{R}^2$

Z_F !!!

$$Z_F = \begin{cases} x_1^2+y_1^2+y_2^2 = 4 \\ y_1 = 1 \end{cases}$$

LET $(\underline{a},\underline{\beta}) \in \mathbb{R}^3$, WHERE $\underline{a} = (0) \in \mathbb{R}^2$

$\underline{\beta} = (\beta_1, \beta_2)$

$(0,\sqrt{3},1) \in Z_F \Leftarrow F(0,\sqrt{3},1) = (0,0) \in \mathbb{R}^2$

$$J_{F((0,\sqrt{3},1))} = \begin{pmatrix} 2x_1 & 2y_1 & 2y_2 \\ 0 & 0 & 1 \end{pmatrix}_{(x_1,y_1,y_2)=(0,\sqrt{3},1)} =$$

$$\begin{array}{cccc} & & y_1 & y_2 \\ = & \Bigg(0 & 2\sqrt{3} & 2 \Bigg) \end{array}$$

$$\det\begin{pmatrix} 2\sqrt{3} & 2 \\ 0 & 1 \end{pmatrix} = 2\sqrt{3} \neq 0$$

$$\begin{vmatrix} 0 & \phi & 1 \end{vmatrix}$$

IS OF DET $\neq 0$!!!

FROM THE THM , WE CAN LOCALLY

$$\begin{cases} y_1 = \varphi_1(x_1) = \sqrt{3 - x_1^2} \\ y_2 = \varphi_2(x_1) \stackrel{?}{=} 1 \end{cases}$$

$\varphi_1, \varphi_2 \in C^{(1)}$

STOP QUESTIONS