LEZ 2 BIS

Monday, February 6, 2023 11:14 AM

BEGIN AT 15.15

RMK "FOR THE SAME REASONS"; FOR $n \in \mathbb{Z}^+$

$$\mu^*(\mathbb{R}^n) = +\infty$$

INDEED, SAY $I_m = \{(x_1, \dots, x_n) \in \mathbb{R}^n ; 0 < x_i < m, \ i = 1, 2, \dots, n\}$

$$I_m \subseteq \mathbb{R}^n \implies$$

$$m^n = \mu^*(I_m) \leq \mu^*(\mathbb{R}^n) \qquad \forall m \in \mathbb{Z}^+$$

$$\Downarrow$$

$$\mu^*(\mathbb{R}^n) = +\infty \qquad !!!$$

PROP (COUNTABLE SUBADDITIVITY PROPERTY)

CONSIDER A FAMILY

$$\{A_k \subseteq \mathbb{R}^n \, ; \, k \in \alpha\}$$ WHERE AT MOST COUNTABLE

THEN

$$\mu^*\Big(\bigcup_{k\in\alpha} A_k\Big) \le \sum_{k\in\alpha} \mu^*(A_k) .$$

PROOF SUPPOSE

$$\sum_{k\in\alpha} \mu^*(A_k) < +\infty ,$$

OTHERWISE THE ASSERTION IS TRIVIAL.

CRUCIAL GIVEN AN ARBITRARY $\varepsilon \in \mathbb{R}^+$

AND GIVEN $k \in \alpha$ THERE EXISTS

$$(\dagger)\begin{cases} \text{A } \underline{\text{LEB. COV.}} \\ \{I_{k_j} \,;\, j \in \mathbb{N}_k\} \text{ OF THE SET } \underline{A_k} \\ \text{SUCH THAT} \\ \displaystyle\sum_{j\in\mathbb{N}_k} \mu(I_{k_j}) < \mu^*(A_k) + \varepsilon/2^k \quad !!! \end{cases}$$

FROM $(\dagger)$

$$\underline{\sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}_k} \mu(I_{k_j})} < \sum_{k\in\mathbb{N}} \left(\mu^*(A_k) + \varepsilon/2^k\right)$$

BUT

$$\bigcup_{k\in\mathbb{N}} \left(\bigcup_{j\in\mathbb{N}_k} I_{k_j}\right) \supseteq \bigcup_{k\in\mathbb{N}} A_k$$

FROM $\underline{\text{CANTOR 1}}$ (!!!), IT FOLLOWS THAT

$$\bigcup_{k\in\mathcal{A}} \{ I_{kj} ; j \in \mathcal{A}_k \} \in \mathcal{J}_{\bigcup_k A_k}$$ ... of $\bigcup_k A_k$.

THEN

$$\mu^*\left(\bigcup_{k\in\mathcal{A}} A_k\right) \leq \sum_{k\in\mathcal{A}} \sum_{j\in\mathcal{A}_k} \mu(I_{kj}) < \sum_{k\in\mathcal{A}} \left(\mu^*(A_k) + \varepsilon/2^k\right) =$$

$$= \sum_{k\in\mathcal{A}} \mu^*(A_k) + \underline{\sum_{k\in\mathcal{A}} \varepsilon/2^k}$$

SUPPOSE $\mathcal{A}$ COUNTABLE $\Longrightarrow$

$$\sum_{k\in\mathcal{A}} \varepsilon/2^k = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \cdot \underbrace{\sum_{k=1}^{\infty} \frac{1}{2^k}}_{\text{HAS SUM } \underline{1}} = \varepsilon$$

HENCE

$$\mu^*\left(\bigcup A_k\right) \leq \sum \mu^*(A_k) + \varepsilon$$

$$\underline{k \in \mathbb{N}} \qquad k \in \mathbb{N}$$

$$\forall \varepsilon \in \mathbb{R}^+$$

$$\Downarrow$$

$$\mu^*\left(\bigcup_{k\in\mathbb{N}} A_k\right) \le \sum_{k\in} \mu^*(A_k) \quad ?!!?$$

<u>QED</u>.

<u>FIRST MAIN CONSEQUENCE</u>

LET $A \subseteq \mathbb{R}^n$ AND ASSUME THAT

A IS AT MOST COUNTA

WHAT ABOUT

$$\mu^*(A) = ???$$

WE HAVE

$$\mu^*(A) = 0 \quad \text{IF } A \text{ AT} \quad \text{COUN}$$

GIVEN ANY SET, WE CAN WRITE

$$A = \bigcup_{a \in A} \{a\}$$

BUT A ATMOST COUNT. $\Rightarrow$ $A = \overline{\bigcup_{a \in A}}^{\text{COUNT}} \{a\}$

WE CAN APPLY THE PREVIOUS THM !!!

TMAT 13

$$\mu^*(A) = \mu^*\Big(\overset{\text{COUNT}}{\bigcup_{x \in A}} \{x\}\Big) \overset{\text{THM}}{\leq} \sum_{x \in A} \underbrace{\mu^*(\{x\})}_{= 0}$$

$$\Downarrow$$

$$\mu^*(A) = 0 \qquad !!!$$

IN PARTICULAR $\mu^{*}(\mathbb{Q}) = 0$