Monday, February 6, 2023 11:14 AM

MAIN QUESTION: IS IT TRUE THAT $A, B \subseteq \mathbb{R}^n$,

$$A \cap B = \emptyset \overset{?}{\Longrightarrow} \mu^*(A \cup B) = \mu^*(A) + \mu^*(B) \quad ???$$

NO

INDEED WE HAVE: $A, B \subseteq \mathbb{R}^n$

RECALL

$$0 \leq d(A,B) = \inf \{ d(a,b) ; a \in A, b \in B \}$$

DISTANCE

PROP IF

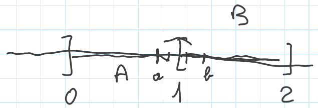
$$d(A,B) > 0 \Longrightarrow \mu^*(A \cup B) = \mu^*(A) + \mu^*(B). \; !!!$$

NOTICE THAT

$$d(A,B) > 0 \underset{\not\Leftarrow}{\Longrightarrow} A \cap B = \emptyset$$

BUT NOT VICEVERSA $(*)$

IN FACT ABOUT $(*)$ LET $A =]0,1[$, $B = [1,2]$



NOW $A \cap B = \emptyset$ BUT $d(A,B) \overset{?}{=} 0$.

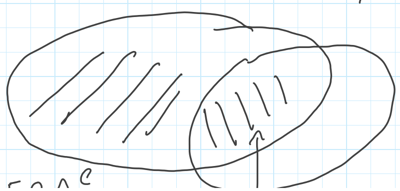
MAIN DEFINITION $A \subseteq \mathbb{R}^n$ (CARATHEODORY CONSTANTIN)

A MEASURABLE $\overset{DEF}{\Longleftrightarrow}$ $\forall E \subseteq \mathbb{R}^n$ WE HAVE

$$\mu^*(E) \overset{!}{=} \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad (\dagger)$$

WHERE $A^c = \mathbb{R}^n \setminus A$

NOTICE E ARBITRARY A FIXED



$E \cap A^c$

$E \cap A$ $\quad \top \; E \cap A$

BY SUBADDITIVITY, IN GENERAL WE HAVE:

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad ???$$

RMK THERE ARE SETS $A \subseteq \mathbb{R}^n$ THAT ARE NOT MEASURABLE SETS !!!

IF A MEASURABLE $\Rightarrow$

$$\underset{\text{MEASURE}}{\mu}(A) = \mu^*(A)$$

RMK BY SYMMETRY OF DEF $(\top)$ $\Rightarrow$

A MEASURABLE $\iff$ A^c MEASURABLE

PROP $A \subseteq \mathbb{R}^n$, $\mu^*(A) = 0$

THEN A MEASURABLE SET

PROOF IN GENERAL, GIVEN $E \subseteq \mathbb{R}^n$

$$\mu^*(E) \leq \underset{\overset{\parallel}{0}}{\mu^*(E \cap A)} + \mu^*(E \cap A^c) \qquad (*)$$

BUT IF $\mu^*(A) \Rightarrow \left(\text{SINCE } E \cap A \subseteq A \overset{\text{MON}}{\Rightarrow} \mu^*(E \cap A) = 0\right)$

THEN (*) $\underline{\mu^*(E) \leq \mu^*(E \cap A^c)}$

BUT $E \cap A^c \subseteq E \Rightarrow$

$\Rightarrow \underline{\mu^*(E \cap A^c) \leq \mu^*(E)}$

$\Rightarrow \mu^*(E) = \mu^*(E \cap A^c)$

HENCE $\forall E \subseteq \mathbb{R}^n$

$$\mu^*(E) = \underset{0}{\underset{\shortparallel}{\mu^*(E \cap A)}} + \mu^*(E \cap A^c) \quad !!!$$

THEN, $\mu^*(A) = 0 \Rightarrow A$ MEASURABLE !!!

AS A CONSEQUENCE:

A COUNTABLE $\Rightarrow$ A MEASURABLE

$\Downarrow$ $\mu^*(A) = 0$ $\nearrow$

$\underline{\text{EX}}$ $\mathbb{Q} \subseteq \mathbb{R}$, $\mu^*(\mathbb{Q}) = 0 \Rightarrow \mathbb{Q}$ MEASURABLE

$\Rightarrow \mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}^c$ MEASURABLE

<u>BREAK</u> <u>QUESTIONS?</u>

BEGIN AGAIN 12.15