Monday, February 13, 2023 10:08 AM

LIMIT THMS FOR NESTED SEQUENCES OF MEASURABLE SETS

PROP. LET $(A_n)_{n \in \mathbb{N}}$, $A_n \subseteq \mathbb{R}^n$, A_n MEASURABLE.

$(A_n)_{n \in \mathbb{N}}$ NESTED INCREASING, THAT IS

$$A_n \subseteq A_{n+1} \quad \forall n \in \mathbb{N}.$$

THEN

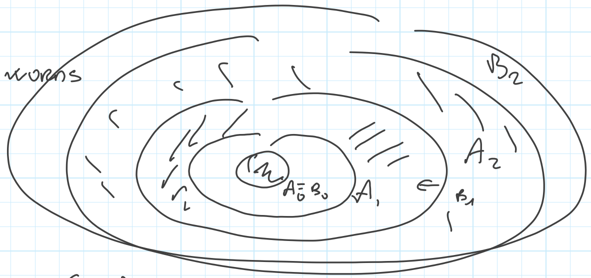
$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n) \quad !!!$$

PROOF. CONSIDER THE AUXILIARY SEQUENCE $(B_n)_{n \in \mathbb{N}}$ S.T.

i) $B_0 = A_0$

ii) $B_{n+1} = A_{n+1} \setminus A_n$

IN PLAIN WORDS



SO $(B_n)_{n \in \mathbb{N}}$ IS A SEQUENCE (+)

OF PAIRWISE DISJOINT SUBSETS & MEASURABLE

NOW $\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} B_n \implies$

$$(*) \quad \mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \mu\left(\bigcup_{n=0}^{\infty} B_n\right) \overset{(+)}{=} \sum_{n=0}^{\infty} \mu(B_n) \quad (**)$$

NOW $(**)$ $\displaystyle\sum_{n=0}^{\infty} \mu(B_n) \overset{DEF}{=} \lim_{n \to \infty} \left(\sum_{k=0}^{n} \mu(B_k)\right)$

SINCE $\bigcup_{k=0}^{n} B_k = A_n \Rightarrow$

$$\sum_{k=0}^{n} \mu(B_k) = \mu(A_n)$$

(*), (**) $\Rightarrow$

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

QED

CONVERSELY, WE HAVE

PROP. LET $(A_n)_{n\in\mathbb{N}}$, $A_n \subseteq \mathbb{R}^n$, A_n MEASURABLE

SUCH THAT NESTED DECREASING, THAT IS

$$A_{n+1} \subseteq A_n \quad \forall n \in \mathbb{N}.$$

HP (SUPPLEMENTARY) $\exists \underline{n} \in \mathbb{N}$ S.T. (#)

$$\mu(A_{\underline{n}}) < +\infty .$$

THEN,

$$\mu\left(\bigcap_{n=0}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

RMK

IF HP (#) IS NEGLECTED, THE THM IS FALSE !!!

FOR EXAMPLE CONSIDER THE SEQUENCE

$$A_n =]n, +\infty[\subseteq \mathbb{R}.$$

NOW $\mu(A_n) = +\infty \quad \forall n \in \mathbb{N}$.

BUT, $\bigcap_{n=0}^{\infty} A_n = \bigcap_{n=0}^{\infty}]n, +\infty[= \emptyset$

$$0 = \mu\left(\bigcap_{n=0}^{\infty} A_n\right) \neq \lim_{n\to\infty} \underbrace{\mu(A_n)}_{=+\infty} = +\infty .$$

THE BOREL σ-ALGEBRA

LET $\mathcal{O}$ = THE FAMILY OF ALL OPEN SUBSETS OF $\mathbb{R}^n$

$\mathcal{C}$ = THE FAMILY OF ALL CLOSED SETS OF $\mathbb{R}^n$

WE RECALL

THM $\mathcal{O}, \mathcal{C} \subseteq \mathcal{L}$ $\leftarrow$ LEBESGUE σ-ALGEBRA OF MEASURABLE SETS.

GENERATED σ-ALGEBRA

RMK LET $\{ \mathcal{E}_i \, ; \, i \in \alpha \}$ BE ANY FAMILY OF σ-ALGEBRAS OF $\mathbb{R}^n$,

THEN $\bigcap_{i \in \alpha} \mathcal{E}_i$ IS A σ-ALGEBRA.

PROOF.

i) $\phi \in \mathcal{E}_i \;\; \forall i \Rightarrow \phi \in \bigcap_{i \in \alpha} \mathcal{E}_i$

ii) $A \in \bigcap_{i \in \alpha} \mathcal{E}_i \Rightarrow A \in \mathcal{E}_i \;\; \forall i \Rightarrow$

$\Rightarrow A^c \in \mathcal{E}_i \;\; \forall i \in \alpha \Rightarrow A^c \in \bigcap_{i \in \alpha} \mathcal{E}_i$.

iii) LET $\{A_k ; k \in \mathcal{J}\}$ AT MOST COUNTABLE

SUCH THAT $A_k \in \bigcap_{i \in \mathcal{U}} \mathcal{E}_i \ \forall k \implies$

$A_k \in \mathcal{E}_i \ \forall i \in \mathcal{U} \implies \bigcup_{k \in \mathcal{J}} A_k \in \mathcal{E}_i \ \forall i \in \mathcal{U}$

$\implies \bigcup_{k \in \mathcal{J}} A_k \in \bigcap_{i \in \mathcal{U}} \mathcal{E}_i$ QED.

LET $\mathcal{J} \subseteq \mathbb{P}(\mathbb{R}^n)$.

$$\mathcal{S}_{\mathcal{J}} \overset{\text{DEF}}{=} \bigcap_{\mathcal{J} \subseteq \mathcal{E}} \mathcal{E}$$

σ-ALGEBRAS

$\Leftarrow$ BY THE RMK, THIS IS A σ-ALGEBRA.

CALLED "THE σ-ALGEBRA GENERATED BY THE FAMILY $\mathcal{J}$"

$\mathcal{S}_{\mathcal{J}}$ IS SUCH THAT $\mathcal{J} \subseteq \mathcal{S}_{\mathcal{J}}$

PRINCIPLE (COMPARISON)

LET $\mathcal{J}, \bar{\mathcal{J}} \subseteq \mathbb{P}(\mathbb{R}^n)$.

IF $\mathcal{J} \subseteq \mathcal{S}_{\bar{\mathcal{J}}} \implies \mathcal{S}_{\mathcal{J}} \subseteq \mathcal{S}_{\bar{\mathcal{J}}}$

IT FOLLOWS, THE IDENTITY PRINCIPLE:

PROP. $\mathcal{I}, \mathcal{J} \subseteq \mathbb{P}(\mathbb{R}^n)$.

If $\quad \mathcal{I} \subseteq \sigma_{\mathcal{J}} \quad$ and $\quad \mathcal{J} \subseteq \sigma_{\mathcal{I}}$

$$\Downarrow$$

$$\sigma_{\mathcal{I}} = \sigma_{\mathcal{J}}.$$

NOW $\quad \mathcal{O}$ = OPEN $\quad \mathcal{C}$ = CLOSED

$$\mathcal{C} \subseteq \sigma_{\mathcal{O}}, \quad \mathcal{O} \subseteq \sigma_{\mathcal{C}}$$

$$\Downarrow$$

$$\sigma_{\mathcal{O}} = \sigma_{\mathcal{C}} \overset{\text{DEF}}{=} \mathcal{B}(\mathbb{R}^n)$$

BOREL σ-ALGEBRA

THE MAIN FACT

$$\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n)$$

$\neq$ PROPER INCLUSION !!!

BREAK QUESTIONS?

BEGIN AGAIN AT 12.15