Monday, February 13, 2023 10:08 AM

BEGIN AT 12.15

EXAMPLE $A = \{ (x;y) \in \mathbb{R}^2 ; (x,y) \notin \mathbb{Q} \times \mathbb{Q} \}$.

IS A A BOREL SET? ($A \in \mathcal{B}(\mathbb{R}^2)$?)

NOW

$$A = \underbrace{\mathbb{R}^2}_{\text{BOREL}} - \underbrace{\{ (x,y) \in \mathbb{R}^2 ; (x,y) \in \mathbb{Q} \times \mathbb{Q} \}}_{\text{IT IS COUNTABLE}}$$

BOREL SET

$\Rightarrow$ IT CAN BE REPRESENTED AS COUNTABLE UNION OF ITS SINGLETON SETS!!!

CLOSED $\Rightarrow$ BORELIAN

THE BOREL σ-ALGEBRA $\mathcal{B}(\mathbb{R})$, $n=1$.

ELEMENTARY QUESTION WHAT IS A OPEN SET $A \subseteq \mathbb{R}$???

WE RECALL (TRIVIAL)

$$A \text{ OPEN} \overset{\text{DEF}}{\Longleftrightarrow} \forall x \in A \ \exists r \in \mathbb{R}^+ \text{ S.T. } I(x,r) \subseteq A.$$

LINDELHÖF LEMMA (WEAK FORM)

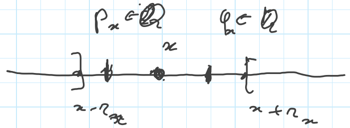
ANY A OPEN, $A \subseteq \mathbb{R}$ CAN BE REPRESENTED AS AT MOST COUNTABLE(!!!) UNION OF LIMITED OPEN INTERVALS !!!

PROOF LET $A \subseteq \mathbb{R}$, A OPEN $\Longleftrightarrow$

$\forall x \in A \quad \exists r \in \mathbb{R}^+$ S.T. (*)

$$I(x, r_x) = \,]x - r_x, x + r_x[\, \subseteq A.$$

NOW



$\exists\, p_x, q_x \in \mathbb{Q}$ SUCH THAT $(*)$

$$x - r_x < p_x < x$$

$$x < q_x < x + r_x$$

FROM $(*)$ AND $(**)$ IT FOLLOWS THAT

$$\forall x \in A \quad \exists\, p_x, q_x \in \mathbb{Q} \text{ s.t. } \quad x \in \,]p_x, q_x[$$

COUNTABLE!! $\swarrow$

$$\Longrightarrow \quad A \subseteq \bigcup_x \,]\underset{\substack{\uparrow \\ \mathbb{Q}}}{p_x}, \underset{\substack{\uparrow \\ \mathbb{Q}}}{q_x}[\qquad \underline{\text{QED}} \text{ !!!}$$

<u>NOW</u>, WE HAVE CONSEQUENCE:

$$\mathcal{B}(\mathbb{R}) = \mathcal{S}_{\mathcal{O}} = \mathcal{S}_{\mathcal{C}}$$

$\mathcal{O}$ OPEN OF $\mathbb{R}$

$\mathcal{C}$ CLOSE OF $\mathbb{R}$.

NOW, $\mathcal{J}$ = OF ALL LIMITED OPEN INTERVALS OF $\mathbb{R}$.

CLEARLY

$$\mathcal{J} \subseteq \mathcal{S}_{\mathcal{O}} = \mathcal{B}(\mathbb{R}) \Longrightarrow$$

$$\mathcal{S}_{\mathcal{J}} \subseteq \mathcal{B}(\mathbb{R}) = \mathcal{S}_{\mathcal{O}} \qquad \underline{\text{TRIVIAL}}. \quad (*)$$

BUT LINDELÖF LEMMA $\Longrightarrow$

BUT, CARATHEODORY LEMMA $\Longrightarrow$

$$\mathcal{O} \subseteq \mathcal{S}_{\zeta} \qquad \text{(NON TRIVIAL PART!!)}$$

$$\Downarrow$$

$$\mathcal{B}(\mathbb{R}) = \mathcal{S}_{\mathcal{O}} \subseteq \mathcal{S}_{\zeta} \qquad (**)$$

THEN, $(*)$ AND $(**)$ $\Longrightarrow$

$$\mathcal{B}(\mathbb{R}) = \mathcal{S}_{\zeta}$$

BREAK QUESTIONS?

BYE BYE !!!