Monday, February 13, 2023 10:09 AM

BEGIN AT 14.10

$\mathcal{B}(\mathbb{R}^n) \overset{\text{DEF}}{=}$ BOREL σ-ALGEBRA $\subsetneq \mathcal{L}(\mathbb{R}^n)$

↑ LEBESGUE σ-ALGEBRA (OF ALL MEASURABLE SETS)

PROP 1 $A \subseteq \mathbb{R}^n$. THEN THERE EXISTS

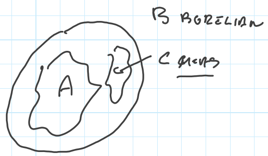
$B \subseteq \mathbb{R}^n$ BOREL SUCH THAT $B \supseteq A$

*) $\mu^*(A) = \mu(B)$

PLUS

**) $\forall C \subseteq B \setminus A$, C MEASURABLE, WE HAVE:

$\mu(C) = 0$.



$\mu^*(A) = \mu(B)$

$\mu(C) = 0$

WE RECALL:

? PROP $A \subseteq \mathbb{R}^n$

$\mu^*(A) \overset{\text{THM}}{=} \inf\{\mu(D);\ D \text{ meas},\ D \supseteq A\}$.

INNER MEASURE OF A: $A \subseteq \mathbb{R}^n$

DEF. $\mu_*(A) = \sup\{\mu(E);\ E \text{ meas},\ E \subseteq A\}$.

CLEARLY, WE $\mu_*(A) \le \mu^*(A)$.

MORE: AN ANALOG OF PROP.

PROP 2 (INNER MEASURES) $A \subseteq \mathbb{R}^n$.

THERE EXISTS A BORELIAN SET $B \subseteq A$

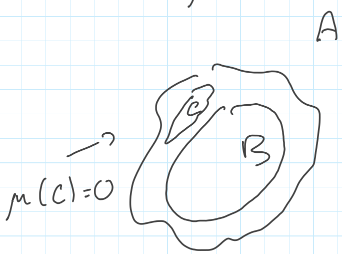
S.T.

+) $\mu_*(A) = \mu(B)$ $\quad B$ BOREL, $B \subseteq A$

PLUS

++) $\forall C \subseteq A \setminus B$, C MEAS WE HAVE:

$$\mu(C) = 0 \quad !!!$$



$\mu_*(A) = \mu(B)$

PLUS

$\forall C \subseteq A \setminus B$, C MEAS

$\mu(C) = 0$

THM 1) A MEASURABLE $\Longrightarrow$

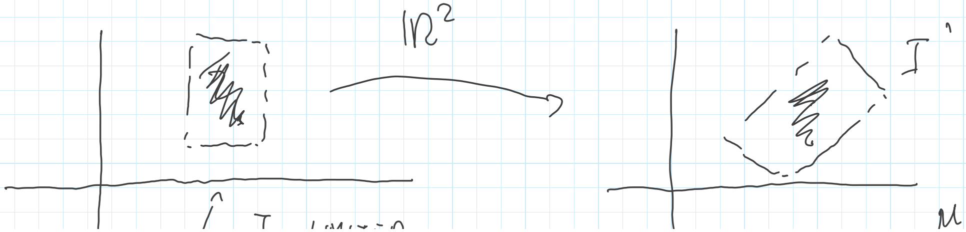
$$\Longrightarrow \mu^*(A) = \mu_*(A) \quad !!!$$

"CONVERSELY"

2) IF $\mu^*(A) = \mu_*(A) < +\infty \Longrightarrow$

$\Longrightarrow$ A MEASURABLE

INVARIANCE PROPERTIES ???



$\mu(I') = ???$

LIMITED
OPEN
INTERVAL

DEF. LET $c \in \mathbb{R}^n$, AND

$$L : \mathbb{R}^n \xrightarrow[1-1]{SU} \mathbb{R}^n$$ LINEAR OPERATOR
INVERTIBLE

AN AFFINE TRANSFORMATION OF

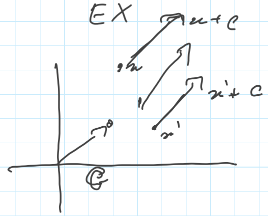
T $\mathbb{R}^n$ ONTO ITSELF IS A MAP

$$x \in \mathbb{R}^n, \quad T(x) = \overbrace{L(x)}^{\text{LINEAR}} + \underbrace{c}$$

LINEAR

T IS INVERTIBLE.

EX



THM IF $A \subseteq \mathbb{R}^n$, A MEAS

1)

$$\Updownarrow$$

$$T[A] \overset{\text{DEF}}{=} \{T(x);\ x \in A\} \text{ MEAS}$$

2)

$$\mu(T[A]) = |\det M_L| \cdot \mu(A)$$

WHERE M_L IS THE MATRIX THAT REPRESENTS

THE "LINEAR PART" $L(x)$

<u>SPECIAL CASE</u> ASSUME THAT

$$|\det M_L| = 1 \iff \det M_L = \pm 1 .$$

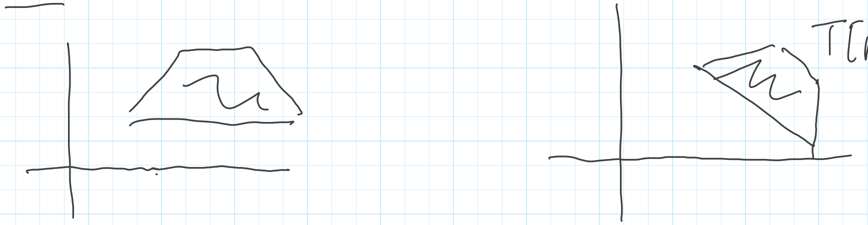
THE TRANSFORATION T IS SAID TO BE

AN <u>ISOMETRY</u>.

<u>COROLLARY</u> IF T IS AN <u>ISOMETRY</u> $\Rightarrow$

$$\underline{\mu(T[A]) = \mu(A)} \quad \forall A \subseteq \mathbb{R}^n, \ A \text{ MEASURABLE}$$

<u>EX</u> A $\mathbb{R}^2$



<u>ISOMETRY</u> ??? T IS SUCH THAT

$\forall x, x' \in \mathbb{R}^n$

$$|\det M_L| = 1 \iff \underline{d(T(x), T(x')) = d(x, x')}$$

" DEF

ISOMETRY

<u>WHAT ABOUT ISOMETRIES?</u>

WE SAY

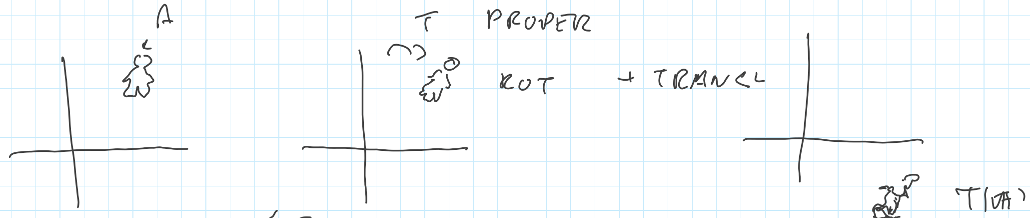
DEF 220

i) IF $\det(M_L) = 1 \overset{\text{DEF}}{\Longrightarrow}$ T PROPER ISOMETRY !!!

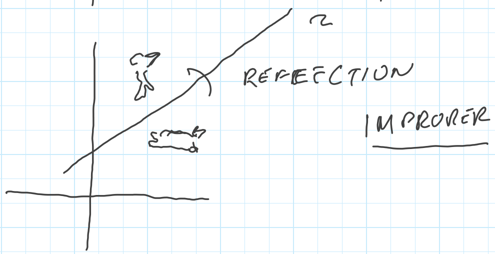
ii) IF $\det(M_L) = -1 \overset{\text{DEF}}{\Longrightarrow}$ T IMPROPER.

PROPER ARE ALL (ONLY) ROTO/TRANSLATIONS !!!

EX



EX



BREAK QUESTIONS?

BEGIN AT 15.15