Monday, February 13, 2023 10:10 AM

BEGIN AT 15.15

MEASURABLE FUNCTIONS (RANDOM VARIABLES, IN PROBABILITY)

$\mathbb{R}$ = THE SET OF REAL NUMBERS

$\overline{\mathbb{R}}$ = " THE EXTENDED REAL NUMBERS SYSTEM "

$= \mathbb{R} \cup \{-\infty, +\infty\}$

"FORMAL" REAL NUMBERS

SUCH THAT

$$-\infty < \alpha < +\infty \quad \forall \alpha \in \mathbb{R}.$$

FURTHERMORE: $\forall \alpha \in \mathbb{R}$

$\alpha - \infty = -\infty$, $\alpha + \infty = +\infty$

AND SO ON...

$\alpha \cdot -\infty = -\infty$, $\alpha \cdot +\infty = +\infty$

AND SO ON... .

THERE IS AN "EXCEPTION":

$-\infty + \infty$ IS NOT DEFINED (IS MEANINGLESS)

WE WILL CONSIDER FUNCTIONS:

$$f : A \longrightarrow \overline{\mathbb{R}}$$

???

$$A \subseteq \mathbb{R}^n$$

WE HAVE:

PROP. LET $f : A \longrightarrow \overline{\mathbb{R}}$, $A \subseteq \mathbb{R}^n$.

THE FOLLOWING ARE EQUIVALENT:

1) $\forall \alpha \in \mathbb{R}$; THE SET

$\{x \in A ; f(x) > \alpha\}$ IS MEASURABLE

2) $\forall \alpha \in \mathbb{R}$, THE SET

$\{x \in A ; f(x) \geq \alpha\}$ IS MEASURABLE

3) $\forall \alpha \in \mathbb{R}$, $\{x \in A;\ f(x) < \alpha\}$ IS MEASURABLE

4) $\forall \alpha \in \mathbb{R}$, $\{x \in A;\ f(x) \leq \alpha\}$ IS MEASURABLE.

DEF

$f : A \longrightarrow \overline{\mathbb{R}}$, $A \subseteq \mathbb{R}^n$

f MEASURABLE $\overset{DEF}{\Longleftrightarrow}$

i) $A \subseteq \mathbb{R}^n$, A MEASURABLE SET IN $\mathbb{R}^n$

ii) 1), 2), 3), 4) ARE TRUE !!!

CONNECTION WITH PROBABILITY (ATTEMPT ???)

Ω A SAMPLE SPACE

X IS RANDOM VARIABLE ON Ω

THIS MEAN THAT

(*) $[X > \alpha]$ IS A EVENT !!!

WHAT THE RELATION BETWEEN (*) AND COND 1) OF MEAS FUNCTS ?

$\underset{\text{EVENT}}{[X > \alpha]} \overset{DEF}{=}$ $\qquad X : \Omega \to \overline{\mathbb{R}}$

$\overset{DEF}{=} \{x \in \Omega;\ X(x) > \alpha\}$ MEASURABLE

COROLLARY $A \subseteq \mathbb{R}^n$, A MEAS.

f MEASURABLE FUNCT $\Longleftrightarrow$

$\forall\, O$ OPEN IN $\mathbb{R}$, $f^{-1}[O] = \{x \in A;\ f(x) \in O\}$ IS MEAS

$\Longleftrightarrow \forall\, C$ CLOSED IN $\mathbb{R}$, $f^{-1}[C] = \{x \in A;\ f(x) \in C\}$ IS MEAS.

COROLLARY $f: A \to \mathbb{R}$, A MEAS.

IF f CONTINUOUS, THEN f IS MEASURABLE.

PROOF RECALL: THE FOLLOWING ARE EQUIVALENT:

1) $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ IS CONTINUOUS ON A

2) $\forall O \subseteq \mathbb{R}$, O OPEN $\exists O_1$ OPEN IN $\mathbb{R}^n$

S.T.

$$f^{-1}[O] = A \cap O_1 \leftarrow \text{OPEN}$$

3) $\forall C \subseteq \mathbb{R}$, C CLOSED $\exists C_1$ CLOSED IN $\mathbb{R}^n$

S.T. $f^{-1}[C] = A \cap \underset{\text{CLOSED}}{C_1}$

NOW ASSUME f CONT ON $A \subseteq \mathbb{R}^n$, A MEAS.

FIX $\alpha \in \mathbb{R}$ CONSIDER 1)

$\{x \in A ; f(x) > \alpha\}$ IS MEASURABLE ?

$$\overset{\shortparallel}{f^{-1}[\,\underset{\text{OPEN}}{]\alpha, +\infty[}\,]} \overset{f \text{ CONT}}{=} \underset{\text{MEAS}}{A} \cap \overset{\text{OPEN} \Rightarrow \text{MEAS}}{O_1}$$

MEASURABLE !!! YES

$\exists O_1$ OPEN IN $\mathbb{R}^n$.

BREAK QUESTIONS ???.

BYEBYE