Saturday, April 23, 2022 11:15 AM

M

MEASURABLE FUNCTIONS

(RANDOM VARIABLE IN PROBABILITY THEORY)

$\overline{\mathbb{R}}$ = EXTENDED REAL NUMBERS

THAT "FORMAL" NUMBERS

$$\overline{\mathbb{R}} = \mathbb{R} \,\dot{\cup}\, \{-\infty, +\infty\}$$

FIELD OF REAL NUMBERS

SUCH THAT $\forall x \in \mathbb{R}$

ALGEBRAIC

i) $x + \infty = +\infty$, $x - \infty = -\infty$

$x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$

<0 $\bar{o}\bar{v}$

WITH THE EXCEPTION:

$+\infty + (-\infty) = \infty - \infty$ IS NOT DEFINED !!!

ORDER PROPERTIES

ii) $\forall x \in \mathbb{R}$

$$-\infty < x < +\infty$$

RMK $f : E \subseteq \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$,

WHERE E IS A MEASURABLE SET IN $\mathbb{R}^n$.

THE FOLLOWING ARE EQUIVALENT:

→ 1) $\forall \alpha \in \mathbb{R}$, THE SET

$\{x \in E ;\ f(x) > \alpha\}$ IS MEASURABLE.

→ 2) $\forall \alpha \in \mathbb{R}$,

$\{x \in E ; f(x) \geq \alpha\}$ IS MEASURABLE.

→ 3) $\forall \alpha \in \mathbb{R}$,

↗ $\{x \in E ; f(x) < \alpha\}$ IS MEASURABLE;

→ 4) $\forall \alpha \in \mathbb{R}$,

$\{x \in E ; f(x) \leq \alpha\}$ IS MEASURABLE.

DEFINITION $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$,

E MEASURABLE.

f IS SAID TO BE A MEASURABLE FUNCTION

$\overset{DEF}{\Longleftrightarrow}$ f SATISFIES 1), 2), 3), 4).

A REMARK

(*) $\{x \in E ; f(x) > \alpha\}$ IS MEASURABLE

READS

PROB. THEORY

X = RANDOM VARIABLE

IS A FUNCTION

$X : \underset{\text{SAMPLE SPACE}}{\Omega} \longrightarrow \overline{\mathbb{R}}$

$X \ (= f)$ (*) BECOMES

$\{x \in E ; f(x) > \alpha\}$ = PROB. THEORY ↙

$\{x \in \Omega ; X(x) > \alpha\} = [X > \alpha]$ IS

MEASURABLE (EVENT)

THM $f: E \subseteq \mathbb{R}^k \to \mathbb{R}$, E MEASURABLE.

IF f CONTINUOUS ON E $\Longrightarrow$

THEN f IS MEASURABLE FUNCTION !!!

PROOF REMEMBER:

f CONTINUOUS ON E $\overset{\text{THM}}{\Longleftrightarrow}$

$\forall\, O$ OPEN IN $\mathbb{R}$ $\exists\, O_1 \subseteq \mathbb{R}^k$, O_1 OPEN

SUCH THAT

$$f^{-1}[O] = \overset{\text{OPEN}}{O_1} \cap E$$

WHERE, BY DEFINITION

$$f^{-1}[O] = \{x \in E;\ f(x) \in O\}$$

"

THE FIBER OF O

"

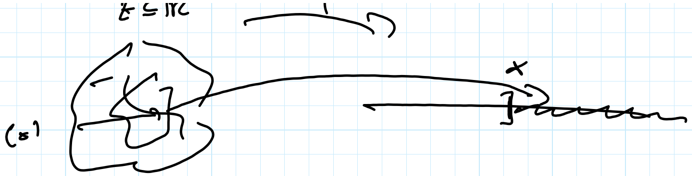
(THE PREIMAGE OF O)

BUT NOW: $\forall\, \alpha \in \mathbb{R}$

(a) $\{x \in E;\ f(x) > \alpha\} \overset{\text{TRIVIALLY}}{=}$ TRUE

$$= f^{-1}[\,]\alpha, +\infty[\,]$$

$E \subseteq \mathbb{R}^k$ f



$$\{x \in E;\ f(x) > \alpha\} =$$

$$= f^{-1}\big(\underbrace{]\alpha, +\infty[}_{\text{OPEN}}\big)$$ CONTINUITY

$\Rightarrow \exists\ \mathcal{O}_1$ OPEN IN $\mathbb{R}^n$ SUCH THAT

$$\rightarrow \{x \in E;\ f(x) > \alpha\} =$$

$$= f^{-1}\big(\underbrace{]\alpha, +\infty[}_{\text{OPEN}}\big) = \overset{\text{MEAS}}{\mathcal{O}_1} \cap \overset{\text{MEAS}}{E}$$ MEASURABLE

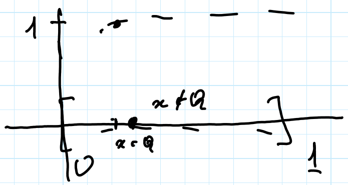
$\mathcal{O}_1$ OPEN $\Downarrow$ MEASURABLE; E MEASURABLE

EXAMPLE

$$f : [0,1] \subseteq \mathbb{R} \rightarrow \mathbb{R}$$

SUCH THAT

$$f(x) = \begin{cases} 0 & \text{IF } x \in [0,1] \setminus \mathbb{Q} \\ 1 & \text{IF } x \in [0,1] \cap \mathbb{Q} \end{cases}$$



NOTICE THAT

f IS DISCONTINUOUS EVERYWHERE!

DIRICHLET "PATOLOGICAL"

FUNCTION"

BUT f IS MEASURABLE.

RECALL $f : [0,1] \to \mathbb{R}$

S.T.

$$f(x) = \begin{cases} 0 & x \in [0,1] \setminus \mathbb{Q} \\ 1 & x \in [0,1] \cap \mathbb{Q} \end{cases}$$

SATISFIES 3)

$\{x \in [0,1] ; f(x) < \alpha\}$ IS MEASURABLE.

CONSIDER THE FOLLOWING CASES.

1) $\alpha > 1 \Longrightarrow$

$\to \{x \in [0,1] ; f(x) < \alpha\} = [0,1]$

CLOSED
$\Downarrow$
MEASURABLE

2) $0 < \alpha \leq 1 \Longrightarrow$

$\{x \in [0,1] ; f(x) < \alpha\} =$

$= \{x \in [0,1] ; f(x) < 1\} =$

$= [0,1] - \mathbb{Q}$ MEASURABLE

CLOSED $\Downarrow$ MEAS

$\mathbb{Q}$ $\Downarrow$ COUNTABLE $\Downarrow$ BOREL SET $\Downarrow$ MEASURABLE

3) $\alpha \le 0$. BUT

$$\{x \in [0,1] \, ; \, f(x) < \alpha\} = \emptyset \text{ MEASURABLE}$$

THEN, THE DIRICHLET function, $f : [0,1] \longrightarrow \mathbb{R}$ IS MEASURABLE.

BREAK QUESTIONS ?

BEGIN AGAIN AT 17.15

MAY WE BEGIN?

(INDICATOR)
CHARACTERISTIC FUNCTIONS

$A \subseteq \mathbb{R}^n$

THE CHARACTERISTIC FUNCT OF A IS

$\rightarrow \chi_A : \mathbb{R}^n \longrightarrow \mathbb{R}$ SUCH THAT

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

CONSISTENCY PROP

PROP

$A \subseteq \mathbb{R}^n$ MEASURABLE (AS A SET)

$\Updownarrow$

χ_A MEASURABLE (AS A FUNCTION)

PROOF $\Downarrow$)

HP $A \subseteq \mathbb{R}^n$ MEAS

NOW

CONSIDER THE SETS

$\alpha \in \mathbb{R}$, $\{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\}$ (∞)

CONSIDER THE CASES

1) $\alpha > 1$ THEN

$(\ast) = \{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\} = \mathbb{R}^n$ MEAS

2) $0 < \alpha \leq 1$ THEN

$\{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\} = A$ MEASURABLE

3) $\alpha \leq 0$

$\{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\} = \emptyset$ MEASURABLE

(DONE !)

$\Uparrow$) USE

χ_A MEASURABLE $\overset{\text{RECALL}}{\Longleftrightarrow}$

$\Longleftrightarrow \forall \alpha \in \mathbb{R}$ $\{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\}$ MEAS

NOW , LET $\alpha = 1$

$\{x \in \mathbb{R}^n ; \chi_A(x) < 1\} =$

$= \{x \in \mathbb{R}^n ; \chi_A(x) = 0\} \overset{\text{DEF}}{=} \mathbb{R}^n \setminus A$

BY IT IS MEAS

THAT IS $\mathbb{R}^n \setminus A$ MEAS $\Longrightarrow$

$\mathbb{R}^n \setminus A = A^c$ MEAS $\Longrightarrow$

$(A^c)^c$ MEASURABLE

$\overset{''}{A}$ MEASURABLE (DONE!!!)

" STABILITY PROPERTIES OF THE CLASS OF MEASURABLE FUNCTIONS "

PROP. $f, g : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$, E MEAS, f, g MEASURABLE (AS FUNCTS)

THEN:

1) $f+g$ MEASURABLE
2) fg MEASURABLE
3) $\lambda \in \mathbb{R}$, λf MEASURABLE.

$\Big\}$ ALGEBRAIC STABILITY

LIMIT OF SEQUENCES OF FUNCTIONS (???)

REMEMBER

$(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$

WE HAVE TWO NOTIONS OF CONVERGENCE

FOR THE SEQUENCE OF FUNCTS $(f_n)_{\in\mathbb{N}}$.

1) POINWISE CONVERGENCE

$f_n \xrightarrow[n\to\infty]{P.W.} f \overset{DEF}{\iff}$

$\Updownarrow$

$\forall \varepsilon \in \mathbb{R}^+ \; \forall x \in E \; \exists n_{\varepsilon, x} \in \mathbb{N}$

$\Downarrow$ (*) SUCH THAT

$$|f(x) - f_n(x)| < \varepsilon \ \ \forall n > n_{\varepsilon,x}$$

WHAT DOES IT MEAN (*)? $\iff$

$$\forall x \in E \ \Big(\forall \varepsilon \in \mathbb{R}^+ \ \exists n_{\varepsilon,x} \in \mathbb{N}$$

SUCH THAT

$$|f(x) - f_n(x)| < \varepsilon \ \ \forall n > n_{\varepsilon,x}\Big)$$

P.W. CONV.

$\Updownarrow$

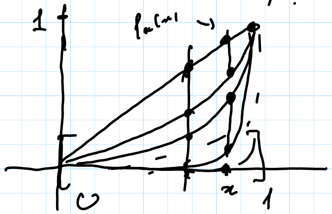
SEQUENCE OF EVALUATIONS

$\forall x \in E$, WE HAVE $f_n(x) \xrightarrow[n \to \infty]{} f(x)$

EX $f_n : [0,1) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ SUCH THAT

$$f_n(x) = x^n .$$

IN A CONCRETE WAY:



$f_1(x) = x$

$f_2(x) = x^2$

$f_3(x) = x^3$

$f_n(x) = x^n$ CONT + LIMITED

IS THIS SEQUENCE POINTWISE CONVERGENT (IF YES, TO WHICH FUNCT?)

YES. IN FACT:

CONSIDER TWO CASES FOR $x \in [0,1]$

1) $0 \le x < 1$ · BUT

CONSIDER

$$f_n(x) \overset{DEF}{=} x^n \xrightarrow[n \to \infty]{} 0 \quad !!!$$

BUT $0 \le x < 1$

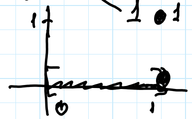
2) $x = 1$ BUT

$f_n(x) = 1^n = 1 \xrightarrow[n\to\infty]{} 1$ YES

CONT + LIM

$f_n \xrightarrow[n\to\infty]{\text{P.W.}} f$ WHERE

$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$



SO, WE DISCOVERED THAT THE CLASS OF CONTINUOUS FUNCTS IS NOT "STABLE" WITH RESPECT TO POINTWISE CONVERGENCE.

2) UNIFORM CONVERGENCE

$(f_n)_{n\in\mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^q \to \mathbb{R}$

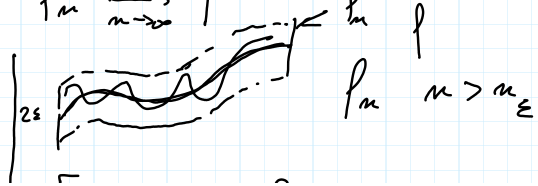
$f_n \xrightarrow[n\to\infty]{\text{UNIF}} f \overset{\text{DEF}}{\iff}$

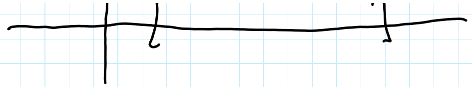
$\forall \varepsilon \in \mathbb{R}^+$ $\exists n_\varepsilon \in \mathbb{N}$ SUCH THAT

$$|f(x) - f_n(x)| < \varepsilon \quad \forall n > n_\varepsilon \quad \forall x \in E$$

WHAT DOES IT MEAN?

$f_n \xrightarrow[n\to\infty]{\text{UNIF}} f$





NOTICE THAT $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$

IS $f_n \xrightarrow[n \to \infty]{P.W.} f$ WHERE

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

BUT

IT IS NOT TRUE

$$f_n \xrightarrow[n \to \infty]{UNIF} f$$

NOT TRUE !!!

IN PLAIN WORDS:

$$f_n \xrightarrow[n \to \infty]{UNIF} f \implies f_n \xrightarrow[n \to \infty]{P.W.} f$$

BUT

$$f_n \xrightarrow[n \to \infty]{P.W.} f \not\Longrightarrow f_n \xrightarrow[n \to \infty]{UNIF} f$$

BREAK QUESTIONS?

BYE