BEGIN AT 12.10

WHAT HAPPENS FOR COUNTABLE FAMILIES OF

OF FUNCTS $f_n : A \subseteq \mathbb{R}^n \to \mathbb{R}$, THAT IS

SEQUENCES OF FUNCTS $(f_n)_{n \in \mathbb{N}}$???

TWO TYPES OF CONVERGENCE

$(f_n)_{n \in \mathbb{N}}$, $f_n : A \subseteq \mathbb{R}^n \to \mathbb{R}$.

1) (POINTWISE CONV)

$(f_n)_{n \in \mathbb{N}}$ IS POINTWISE CONV TO $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$

IF AND ONLY IF

(*) $\forall x \in A \left(\forall \varepsilon \in \mathbb{R}^+ \ \exists \nu_{x,\varepsilon} \in \mathbb{N} \text{ SUCH THAT } |f_n(x) - f(x)| < \varepsilon \ \forall n > \nu_{x,\varepsilon} \right)$

$\Updownarrow$

$\forall x \in A \left(f_n(x) \xrightarrow[n \to \infty]{} f(x) \right)$

2) (UNIFORM CONVERGENCE)

$(f_n)_{n \in \mathbb{N}}$ IS UNIF. CONVERGENT TO $f : A \to \mathbb{R}$

IF AND ONLY IF

(**) $\forall \varepsilon \in \mathbb{R}^+ \ \exists \nu_\varepsilon \in \mathbb{N}$ S.T.

$|f_n(x) - f(x)| < \varepsilon \ \forall n > \nu_\varepsilon \ \forall x \in A$!!!

$f_n \xrightarrow{\text{UNIF}} f \implies f_n \xrightarrow{\text{PW}} f$

$\lim_{n\to\infty}$ … $\lim_{n\to\infty}$

$\Leftarrow$???

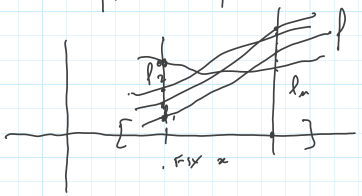
IN PLAIN WORDS FOR SEQUENCES OF FUNCTS

$f_n : A \subseteq \mathbb{R} \to \mathbb{R}$:

PW. CONV "VERTICAL INTERPRETATION"

$f_n \xrightarrow[n\to\infty]{PW} f$

$f_n(x) \xrightarrow[n\to\infty]{} f(x)$ in $\mathbb{R}$!!



CONCRETE VERY USEFUL EXAMPLE

LET $f_n : [0,1] \subseteq \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{Z}^+$

S.T. $f_n(x) = x^n$

$f_1(x) = x$

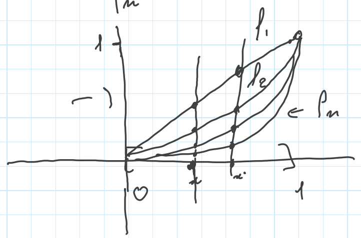
$f_2(x) = x^2$

…

$f_n(x) = x^n$

⋮

SO



IS $(f_n)_{n\in\mathbb{Z}^+}$ PW CONV ? TO WHICH FUCT ?

YES

TO WIT : IF $0 \le x < 1$

$(f_n(x))_{n\in\mathbb{Z}^+} = (x^n)_{n\in\mathbb{Z}^+} \xrightarrow[n\to\infty]{} 0$ SINCE $0 \le x < 1$

AND IF $x = 1$

$(f_n(x) = (1^n)_{n\in\mathbb{Z}^+} \xrightarrow[n\to\infty]{} 1$. !!!

$f_n \xrightarrow[n\to\infty]{PW} f$ WHERE $f(x) \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

FROM THIS WE HAVE:

$(f_n)_{n\in\mathbb{Z}}$, $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ POLYNOMIALS

$\Downarrow$

$(+) \begin{cases} \text{CONTINUOUS} \\ + \\ \text{LIMITED ON } [0,1] \end{cases}$

PLUS, WE DISCOVERED THAT

$f_n \xrightarrow[n\to\infty]{PW} f$ WHERE

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

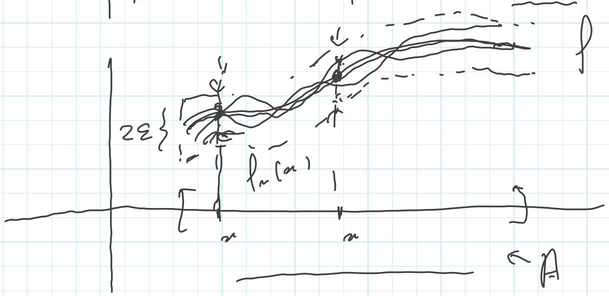
HENCE, f IS DISCONTINUOUS AT $x=1$!!!

INTERPRETATION OF UNIF. CONVERGENCE "HORIZONTAL"

RECALL $f_n \xrightarrow[n\to\infty]{UNIV} f \overset{DEF}{\iff}$

$$\forall \varepsilon \in \mathbb{R}^+ \ \exists \nu_\varepsilon \in \mathbb{N} \text{ S.T.}$$

$$|f_n(x) - f(x)| < \varepsilon \ \forall n > \nu_\varepsilon, \ \forall x \in A \ !!!$$



FIX $\varepsilon \in \mathbb{R}^+$

LET $n > \nu_\varepsilon$

FOR ANY POINT $x \in A$!!

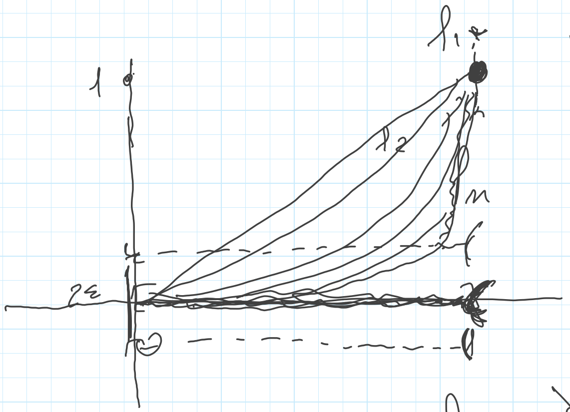
AS AN APPL. IS IT TRUE THAT

$f_n(x) = x^n$ ... $[0,1]$

$f_n(x) = x^n \quad , \quad x \in [0,1]$

IS SUCH THAT $\quad f_n \xrightarrow[n \to \infty]{\text{UNIF}} f \quad$???

WHERE $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$



$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 \text{ !!! } & x = 1 \end{cases}$

FIX $\varepsilon \in \mathbb{R}^+$

SO OUR $\quad f_n \cancel{\xrightarrow[n \to \infty]{\text{UNIF}}} f$

BUT $\quad (f_n)_{n \in \mathbb{N}} \quad f_n$ LIMITED CONTINUOS

$+$

$f_n \xrightarrow[n \to \infty]{\text{UNIF}} f \quad \Longrightarrow \quad f \; \underline{\frac{\text{CONT}}{\text{+ LIMITED}}}$!!!