Monday, February 13, 2023 10:11 AM

BEGIN AT 11.10

LET $(f_n)_{n \in \mathbb{Z}^+}$, $f_n : E \subseteq \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$

SET

i) $(\inf_n f_n)(x) \overset{DEF}{=} \inf_n \{ f_n(x) ; n \in \mathbb{Z}^+ \}$ $\quad x \in E$

ii) $(\sup_n f_n)(x) \overset{DEF}{=} \sup_n \{ f_n(x) ; n \in \mathbb{Z}^+ \}$ $\quad x \in E$

THM LET $f_n : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ BE MEASURABLE $\forall n \in \mathbb{Z}^+$. THEN

$\inf_n f_n$ AND $\sup_n f_n$ ARE MEASURABLE.

AS SPECIAL CASES, WE RECALL:

LET $(f_n)_{n \in \mathbb{N}}$ BE SEQUENCE, $f_n : E \subseteq \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$.

*) $\min_n \lim f_n \overset{DEF}{=} \sup_n \left(\inf_{k \geq n} f_k \right)$

**) $\max_n \lim f_n = \inf_n \left(\sup_{k \geq n} f_k \right)$

COROLLARY IF f_n MEASURABLE $\forall n$, THEN

$\min_n \lim f_n$ AND $\max_n \lim f_n$ ARE MEASURABLE.

RECALL, $(f_n)_{n \in \mathbb{Z}^+}$ IS POINTWISE CONVERGENT

$\Updownarrow$

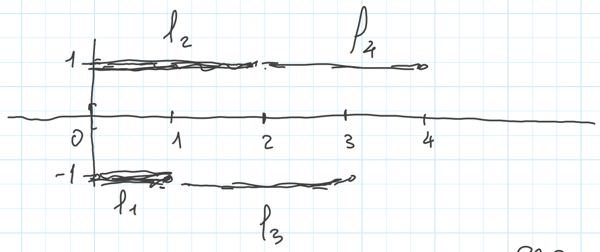
PW LIMIT

$\min_n \lim f_n = \max_n \lim f_n \left(= \lim_{n \to \infty} f_n \right)$

EXAMPLE LET $(f_n)_{n \in \mathbb{Z}^+}$ S.T.

$$f_n = (-1)^n \chi_{[0,n]} : [0, +\infty[\longrightarrow \mathbb{R}.$$

SO



WHAT IS $\liminf_n f_n$??? $(+)$

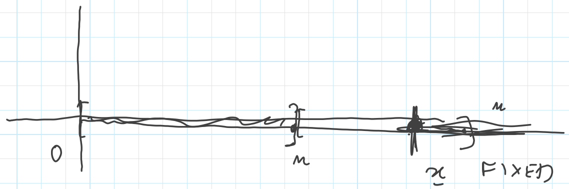
WE RECALL THAT

$$\left(\liminf_n f_n\right)(x) \overset{DEF}{=} \liminf_n f_n(x) \qquad \forall x \in [0,+\infty[$$

TO GIVE ANSWER TO $(+)$ WHAT WE HAVE TO DO?

FIX A POINT $x \in [0, +\infty[$ AND STUDY

$\longrightarrow \left(f_n(x)\right)_{n\in\mathbb{Z}^+}$ . . . . . IN PLAIN WORDS . . .



(**) IF $n < x$, THEN

$$f_n(x) = (-1)^n \chi_{[0,n]}(x) = 0$$

THEN

$$f_n(x) = (\overbrace{0, 0, \dots, 0}^{n<x}, \underbrace{(-1)^n, (-1)^{n+1} \dots}_{n \geq x}) \qquad (++)$$

BUT FOR $n \geq x$, WE HAVE

$$f_n(x) = (-1)^n \chi_{[0,n]}(x) = (-1)^n$$

THEN $(\#)$ IS $(0, 0, \ldots, 0, -1, 1, -1, 1, -1, \ldots$

THEN, STUDY

$$\min\lim_n \left(\overline{0, \underline{0, \ldots, \overline{0, -1, 1, -1, 1, \ldots}}}\right) \overset{DEF}{=}$$

$$\min\lim_n f_n(x) = \sup_n \left(\inf_{k \geq n} f_k(x)\right)$$

$$= \sup_n \left(-1, -1, -1, \ldots, -1 \ldots, -1 \right) = -1$$

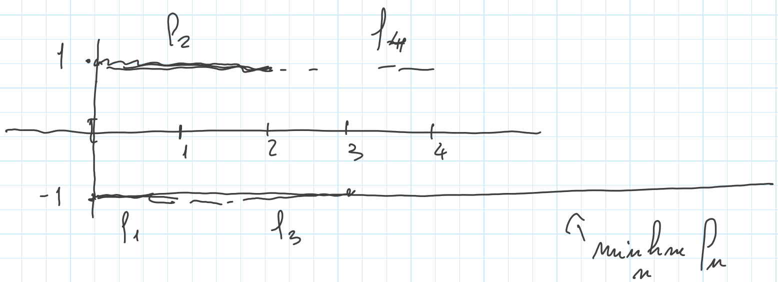
THEN, SINCE $x \in [0, +\infty[$ IS FIXED (ARBITRARY)

WE PROVED:

$$\min\lim_n f_n(x) = -1 \qquad \forall x \in [0, +\infty[$$

THEN

$$\left(\min\lim_n f_n\right)(x) = -1 \qquad \forall x \in [0, +\infty[$$



SIMILARLY, FIXED $x \in [0, +\infty[$ STUDY

$$\max\lim_n f_n(x) = \inf_n \left(\sup_{k \geq n} f_k(x)\right)$$

WE RECALL THAT

$$(f_n(x)) = \left(\overline{0, \underline{0, \ldots, 0, -1, 1, -1, 1, \ldots}}\right)$$

HENCE

$$\left(\sup_{k \geq n} f_k(x)\right)_n = (1, 1, 1, \ldots, 1, \ldots 1, \ldots)$$

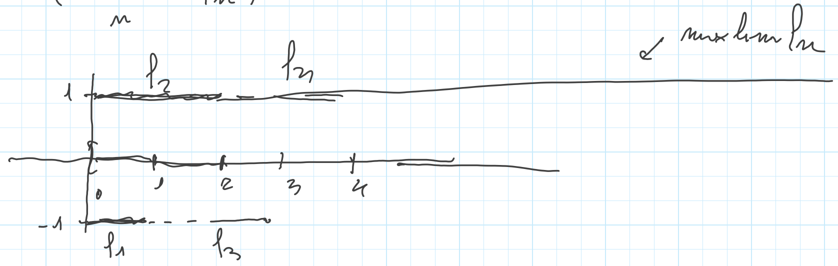
THEN

$$\max\lim_n f_n(x) = \inf_n \left(\overbrace{\sup_{k \geq n} f_k(x)}\right) = \inf_n (1, 1, \ldots, 1, \ldots) = 1$$

TRUE $\forall x \in [0, +\infty[$

$\Downarrow$

$$\left(\max\lim_n f_n\right)(x) = 1 \qquad \forall [0, +\infty[$$



SINCE $\min\lim_n f_n \equiv -1 \underset{\neq}{<} \max\lim_n f_n \equiv 1$

$\Downarrow$

OUR SEQUENCE $f_n = (-1)^n \chi_{[0,n]}$

IS NOT POINTWISE CONVERGENT.

BREAK QUESTIONS?

BEGIN 12.05