Monday, February 13, 2023 10:12 AM

BREAK

QUESTIONS?

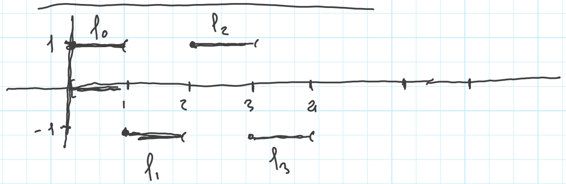
BEGIN AT 12.05

EXAMPL BIS $(f_n)_{n\in\mathbb{N}}$ WHERE

$$f_n = (-1)^n \chi_{[n,n+1[} : [0,+\infty[\longrightarrow \mathbb{R}.$$

STUDY $\min_n \lim f_n$, $\max_n \lim f_n$ AND $\lim_{n\to\infty} f_n$ (EXISTS???)

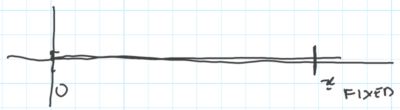
THE SITUATION IS THE FOLLOWING:



FIX $x \in [0,+\infty[$ AND STUDY

$(f_n(x))_{n\in\mathbb{N}}$ - - - - -

NOW, REMARK:



NOTICE THAT, $\exists!\ n_x \in \mathbb{N}$ SUCH THAT

$\underset{\text{FIXED}}{x} \in [n_x, n_x+1[$

THEN

$$(f_n(x))_{n\in\mathbb{N}} = (0, 0, \ldots, 0, \overset{\swarrow\, n_x \text{ POSITION}}{(-1)^{n_x}}, 0, \ldots\ \ldots) \quad x \text{ FIXED}$$

THEN $f_n(x) = (0, \ldots, 0, \pm 1, 0, \ldots, 0 \ldots)$

IS CONVERGENT AND

$\lim_{n\to} f_n(x) = 0$, $\forall x \in [0,+\infty]$

THEN $f_n \xrightarrow[n\to\infty]{\text{P.W.}} f \equiv 0$!!! QED.

LITTLEWOOD "SLOGAN" ANY SEQUENCE

$(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$ MEASURABLE,

WITH $\mu(E) < +\infty$

THAT IS POINTWISE CONVERGENT

$\left(f_n \xrightarrow[n \to \infty]{P.W.} f \text{ MEASURABLE} \right)$

IS "ALMOST UNIFORM CONVERGENT" (???)

FIRST FORMULATION

EGOROFF THM LET $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$

MEASURABLE WITH $\mu(E) < +\infty$.

IF $f_n \xrightarrow[n \to \infty]{P.W.} f$ THEN

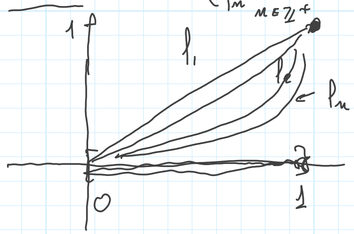
$\forall \delta \in \mathbb{R}^+$ $\exists B \subseteq E$ SUCH THAT

i) $\mu(B) < \delta$

PLUS

ii) $f_n \xrightarrow[n \to \infty]{UNIF} f$ ON THE SUBSET $E \setminus B$.

EXAMPLE LET $(f_n)_{n \in \mathbb{Z}^+}$, $f_n(x) = x^n : [0,1] \to \mathbb{R}$



$f_n \xrightarrow[n \to \infty]{PW} f$ WHERE

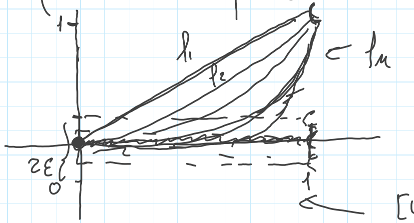
$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

FIRST IS TRUE THAT IF WE REMOVE $B = \{1\}$

(NOTICE $\mu(\{1\}) = 0$) THE SEQUENCE

IS UNIFORM CONV TO f ON $[0,1] \setminus \{1\}$??? NO!!!

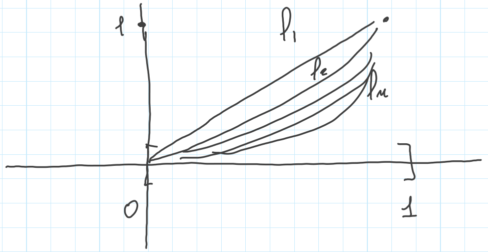
(NOTICE THAT $f : [0,1] \setminus \{1\} = [0,1[$ IS IDENTICALLY ZERO)



$\forall \varepsilon \in \mathbb{R}^+$

$[0,1[= [0,1] \setminus \{1\}$

WHAT INDEE HAPPENS IS THE FOLLOWING:

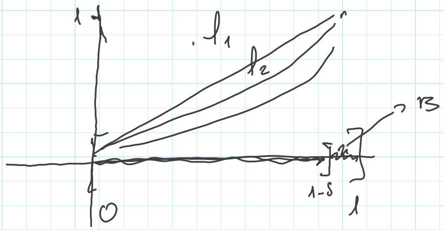


LET $\delta \in \mathbb{R}^+$

$\delta < 1$

NOW, SET $B =]1-\delta, 1]$ $(\mu(B) = \delta)$

AND REMOVE B FROM THE ORIGINAL DOMAIN $[0,1]$

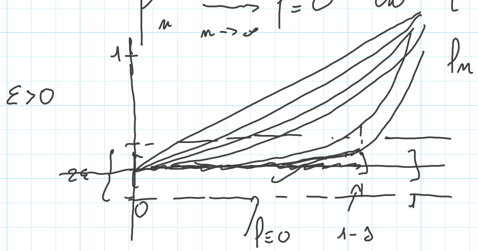


SO

$[0,1] \setminus B = [0, 1-\delta]$

IS IT TRUE THAT

$f_n \xrightarrow[n\to\infty]{\text{UNIF}} f \equiv 0$ ON $[0, 1-\delta] = [0,1] \setminus B$??? (*)



IS IT TRUE

FOR n SUFF. LARGE

THE GRAPH ON f_n

LIES WITHIN THE STRIP ??

FOMALLY $\sup_{x \in [0,1-\delta]} f_n(x) = (1-\delta)^n$ WITH $1-\delta < 1$

$\forall\, \varepsilon \in \mathbb{R}^+ \quad \exists\, n_\varepsilon : \quad (1-\delta)^n < \varepsilon \quad \forall\, n > n_\varepsilon .$

HENCE FOR $n > n_\varepsilon$

$$f_n(x) < \varepsilon \qquad \forall\, x \in [0, 1-\delta]$$

$$\Updownarrow$$

$$|f_n(x) - \underset{\overset{\shortparallel}{0}}{f(x)}| < \varepsilon \quad \forall\, n \geq n_\varepsilon \quad \forall\, x \in \underline{[0, 1-\delta]}$$

$$\Updownarrow \text{ DEF}$$

$$f_n \xrightarrow[n\to\infty]{\text{UNIF}} f \equiv 0 \quad \text{ON} \quad [0, 1-\delta].$$

<u>LITTLEWOOD LEMMA</u> $(f_n)_{n \in \mathbb{N}}$ SEQUENCE OF MEASURABLE FUNCTS

WITH $\mu(E) < \infty$, $f_n : E \subseteq \mathbb{R}^m \to \overline{\mathbb{R}}$

$$f_n \xrightarrow[n\to\infty]{\text{PW}} f \quad \text{ON} \quad E \subseteq \mathbb{R}^n.$$

HENCE FOR $\varepsilon \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$

EXIST $B \subseteq E$ WITH $\mu(B) \leq \eta$ AND $n_{\varepsilon,\eta}$

AND $\underline{|f_n(x) - f(x)| < \varepsilon}$

FOR $n > n_{\varepsilon,\eta}$ and $x \in E \setminus B$ $\quad (\mu(B) \leq \eta)$

<u>STOP</u> QUESTIONS?

BYEBYE