Monday, February 13, 2023 10:12 AM

THE LEBESGUE INTEGRAL THEORY

1) SIMPLE FUNCTIONS
2) $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ LIMITED AND $\mu(E) < +\infty$
3) f MEASURABLE NON/NEGATIVE
4) f GENERAL MEASURABLE FUNCTS.

SIMPLE FUNCTIONS

$\varphi : E \subseteq \mathbb{R}^n \to \mathbb{R}$, E MEAS, $\mu(E) < +\infty$

IS SAID TO SIMPLE

IF AND ONLY IS

(*) $\varphi = \sum_{i=1}^{n} c_i \chi_{E_i}$, WHERE $c_i \in \mathbb{R}$

i) $E_i \subseteq E$

ii) E_i MEASURABLE $\forall i = 1, 2, \dots, n$

(iii) $\mu(E_i) < \infty$, $i = 1, 2, \dots, n$).

COND ii) $\Longrightarrow$ χ_{E_i} MEAS. FUNCT $\forall i = 1, 2, \dots, n$

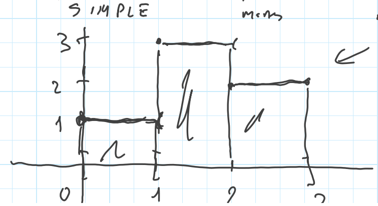
$\Downarrow$

φ SIMPLE $\Longrightarrow$ φ MEASURABLE FUNCT.

THE PRESENTATION (*) IS CLEARLY NOT UNIQUE !!!

EX. LET $\varphi : [0,3] \to \mathbb{R}$, WHERE

(•) $\underset{\text{SIMPLE}}{\varphi} = 1 \cdot \overset{\text{MEAS}}{\underset{\text{MEAS}}{\chi_{[0,2[}}} + 2 \overset{\text{MEAS}}{\underset{\text{MEAS}}{\chi_{[1,3]}}}$



THE AREA OF THIS PORTION IS $\int_E \varphi$

NB.

(**) $\varphi = 1 \cdot \chi_{[0,1[} + 3\chi_{[1,2[} + 2\chi_{[2,3]}$

AND (*) ≠ (**)

CANONICAL PRESENTATION OF A SIMPLE FUNCT φ

LET $\varphi = \sum_{i=1}^{n} e_i \cdot \chi_{E_i}$ (†) IS SAID TO BE CANONICAL

IF AND ONLY IF

i) $E_i \cap E_j = \emptyset$ IF $i \neq j$

ii) $e_1, e_2, \dots, e_n \neq 0$ AND

$e_i \neq e_j$ IF $i \neq j$.

THE CANONICAL PRESENTATION (†) IS

CLEARLY UNIQUE !!!

DEF THE LEBESGUE INTEGRAL $\int_E \varphi$

IS, BY DEFINITION,

$$\int_E \varphi = \sum_{i=1}^{n} e_i \cdot \mu(E_i) \qquad \text{DEF } (*)$$

IN THE PREVIOUS EXAMPLE: $\varphi : [0,3] \to \mathbb{R}$

SUCH THAT

$$\varphi = 1 \cdot \chi_{[0,2[} + 2 \cdot \chi_{[1,3]} \leftarrow \text{NOT CANONICAL}$$

$$= 1 \cdot \chi_{[0,1[} + 3\chi_{[1,2[} + 2\chi_{[2,3]} \leftarrow \text{CANONICAL REPRESENTATION}$$

BY THE PREVIOUS DEF $\Rightarrow$

$$\int_{[0,3]} \varphi = 1 \cdot \mu([0,1[) + 3 \cdot \mu([1,2[) + 2\mu([2,3]) =$$

$$= 1 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 = 6 \quad !!!$$

A SIMPLE ARGUMENT SHOWS:

PROP 1 (LINEARITY) LET $\varphi, \psi : E \subseteq \mathbb{R}^n \to \mathbb{R}$ SIMPLE, $a, b \in \mathbb{R}$. CLEARLY:

i) $a\varphi + b\psi$ IS SIMPLE

ii)!!! $\int_E (a\varphi + b\psi) = a\int_E \varphi + b \cdot \int_E \psi$. LINEARITY.

PROP 1 IMPLIES:

LET $\varphi = \sum_{j=1}^{m} c_j \cdot \chi_{A_j}$ BE A NON CANONICAL REPRESENTATION

NOW, LINEARITY PROP IMPLIES:

$$\int_E \varphi = \int \left(\sum_{j=1}^{m} c_j \cdot \chi_{A_j}\right) \overset{\text{LINEARITY}}{=} \sum_{j=1}^{m} c_j \underbrace{\int \chi_{A_j}}_{\mu(A_j)} =$$

$$= \sum_{j=1}^{m} c_j \mu(A_j)$$

IN THE PREVIOUS EX:

$$\varphi = 1 \cdot \chi_{[0,2[} + 2\chi_{[1,3]}$$

HENCE

$$\int_{[0,3]} \varphi \overset{\text{THM}}{=} 1 \cdot \underbrace{\mu([0,2[)}_{2} + 2\underbrace{\mu([1,3])}_{2} =$$

$$= 1 \cdot 2 + 2 \cdot 2 = 6 \quad \text{AS EARLIER!!}$$

PROP 2 LET $\varphi, \psi : E \subseteq \mathbb{R}^n \to \mathbb{R}$ BE SIMPLE FUNCTS.

ASSUME THAT

$$\varphi \leq \psi \text{ A.E.} \qquad (*)$$

(REMEMBER : $(*) \iff \mu(\{x \in E ; \varphi(x) > \psi(x)\}) = 0$)

THEN $\int_E \varphi \leq \int_E \psi$!!!

EX LET $\varphi : [0,1] \longrightarrow \mathbb{R}$ S.T.

$$\varphi(x) = \begin{cases} 1 & x \in [0,1] \setminus \mathbb{Q} \\ 0 & x \in [0,1] \cap \mathbb{Q} \end{cases}$$

$\leftarrow$ DIRICHLET PATOLOGICAL FUNCT

$\Downarrow$

IT IS EVERYWHERE DISCONTINUOUS ON $[0,1]$

$\Downarrow$

IT HAS NOT INTEGRAL IN THE SENSE OF RIEMANN !!!

CLEARLY, $\varphi = \chi_{\underbrace{[0,1]\setminus\mathbb{Q}}_{\text{MEAS}}}$ MEAS

$\uparrow$ SIMPLE

HENCE, $\int_{[0,1]} \varphi = 1 \cdot \mu(\underbrace{[0,1] \setminus \mathbb{Q}}_{= 1}) = 1$

BREAK QUESTIONS?

BEGIN AGAIN AT 15.10