Monday, February 13, 2023 10:13 AM

STEP 2 (CRUCIAL ONE)

LET $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ LIMITED

AND

$$\mu(E) < +\infty .$$

UPPER LEBESGUE INTEGRAL

UPPER $$\overline{\int_E} f \overset{DEF}{=} \inf_{\substack{\varphi \geq f \\ \varphi \text{ SIMPLE}}} \int_E \varphi \quad \leftarrow \text{(DEFINED BY STEP 1)}$$

LOWER $$\underline{\int_E} f \overset{DEF}{=} \sup_{\substack{\psi \leq f \\ \psi \text{ SIMPLE}}} \int \psi \quad \leftarrow \text{DEFINED BY STEP 1)}$$

CLEARLY $$\underset{\text{SIMPLE}}{\psi} \leq f \leq \underset{\text{SIMPLE}}{\varphi} \implies$$

$$\implies \varphi \geq \psi \overset{\text{MONOTONICITY}}{\implies}$$

$$\overline{\int_E} f = \inf_{\substack{\varphi \geq f \\ \varphi \text{ SIMPLE}}} \int \varphi \geq \sup_{\substack{\psi \leq f \\ \psi \text{ SIMPLE}}} \int \psi = \underline{\int_E} f$$

IN PLAIN WORDS:

$$\text{LOWER} \underline{\int_E} f \leq \text{UPPER} \overline{\int_E} f \quad !!!$$

(*) WE SAY THAT OUR FUNCTION f IS INTEGRABLE IN THE SENSE OF LEBESGUE IF AND ONLY IF

$$\text{LOWE} \underline{\int_E} f \overset{!}{=} \text{UPPER} \overline{\int_E} f .$$

RMK WHY IS STEP 2) WE REQUIRED:

f IS LIMITED ???

LET US RECALL:

UPPER $$\overline{\int_E} f \overset{\text{def}}{=} \inf_{\substack{\varphi \geq f \\ \varphi \text{ simple}}} \int \varphi$$

IF f WERE NOT UPPER LIMITED, DOES THERE EXIST A SIMPLE FUNCT φ S.T. $\varphi \geq f$? NO

LOWER $$\underline{\int_E} = \sup_{\substack{\psi \leq f \\ \psi \text{ simple}}} \int \psi$$

IF f NOT LOWER LIMITED, THERE WERE NO ψ SIMPLE SUCH THAT $\psi \leq f$ ...

THM !!! $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ LIMITED + $\mu(E) < +\infty$.

WE HAVE:

$$\underline{\int_E} f \overset{\text{DEF}}{=} \sup_{\substack{\psi \leq f \\ \psi \text{ simple}}} \int \psi = \inf_{\substack{\varphi \geq f \\ \varphi \text{ simple}}} \int \varphi = \overline{\int_E} f$$

IF AND ONLY IF

f IS A MEASURABLE FUNCTION !!!

PROP $f, g : E \subseteq \mathbb{R}^n \to \mathbb{R}$ LIMITED, $\mu(E) < +\infty$

f, g MEASURABLE $\Leftrightarrow$ LEBESGUE INTEGRABLE

i) $\forall a, b \in \mathbb{R}$ WE HAVE

$$\int_E (af + bg) = a \int_E f + b \int_E g \qquad \text{LINEARITY}$$

ii) MONOTONICITY IF $f \leq g$ A.E.

THEN $$\int_E f \leq \int_E g$$

iii) SPECIAL CASE OF ii) IS

$$\left| \int_E f \right| \leq \int_E |f|$$

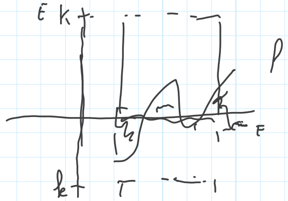
iv) LET $k, K \in \mathbb{R}$ BE SUCH THAT

$$k \leq f(x) \leq K \quad \forall x \in E.$$

THEN

$$k \cdot \mu(E) \leq \int_E f \leq K \cdot \mu(E).$$

TRIVIAL EXAMPLE



v) (ADDITIVITY) LET $E_1, E_2 \subseteq E$, SUCH THAT

$$E_1 \cap E_2 = \emptyset \quad \text{AND} \quad E_1 \dot{\cup} E_2 = E.$$

THEN

$$\int_E f = \int_{E_1} f + \int_{E_2} f$$

NEXT AIM: TO MAKE CLEAR RELATIONS

BETWEEN RIEMANN INTEGRAL AND LEBESGUE INTEGRAL

RECALL RIEMANN INTEGRAL FOR A FUNCTION

$$f : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}. \qquad (*)$$

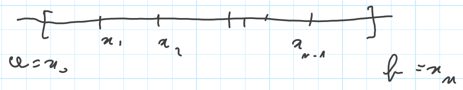
SO, RIEMANN INTEGRAL (SURVEY) FOR FUNCTS OF TYPE $(*)$

GIVEN AN INTERVAL $[a,b] \subseteq \mathbb{R}$

A SUBDIVISION OF $[a,b]$ IS A $(n+1)$-TUPLE

$$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

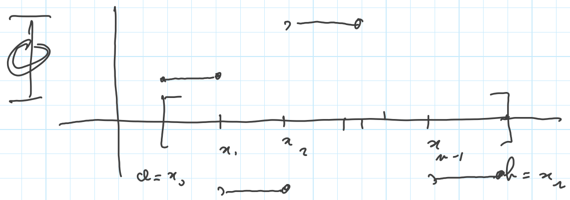
EX



A STEP FUNCTION Φ ON $[a,b]$

IS A FUNCTION OF THE FORM

$$\Phi = c_0 \cdot \chi_{[x_0,x_1]} + \sum_{i=2}^{n} c_i \, \chi_{]x_{i-1},x_i]}$$



NOTICE THAT IF

Φ IS STEP FUNCT $\Longrightarrow$ Φ IS SIMPLE FUNCT !!!

$\not\Longleftarrow$

GIVEN f LIMITED, $f:[a,b] \longrightarrow \mathbb{R}$

WHAT IS THE RIEMANN INTEGRAL

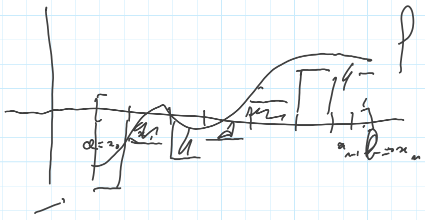
$$R\int_a^b f \quad ???$$

LOWER RIEMANN INTEGRAL:

$$R\underline{\int_a^b} f \overset{DEF}{=} \sup_{\substack{\Phi \le f \\ \Phi \text{ IS STEP}}} \int \Phi \qquad !!!$$

EX

$\underline{\Phi}$

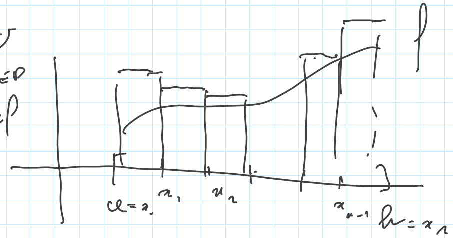


WHERE $\int_a^b \underline{\Phi} = e_0 (x_1 - x_0) + \sum_{i=2}^{n} e_i (x_i - x_{i-1})$

$$R\int_a^b f \overset{\text{DEF}}{=} \inf_{\substack{\overline{\psi} \geq f \\ \overline{\psi} \text{ STEP}}} \int \overline{\psi}$$

$\overline{\psi}$ STEP

$\overline{\psi} \geq f$



$\int \psi$

NOW, DEF:

f ADMITS <u>INTEGRAL IN THE SENSE OF RIEMANN</u>

IF AND ONLY IF

$$R\underline{\int_a^b} f = R\overline{\int_a^b} f \quad !!!$$

QUESTIONS?

BYE BYE