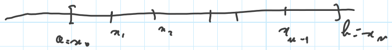
Monday, February 13, 2023 10:13 AM

RIEMANN INTEGRAL FOR (REAL)

$f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ LIMITED.

GIVEN A SUBDIVISION

$x_0 = a < x_1 < \dots < x_{n-1} < x_n = b$



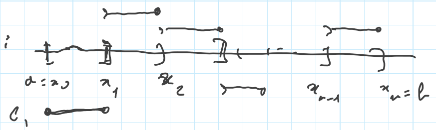
A STEP FUNCTION IS A FUNCTION OF THE FORM

$$\Phi = c_1 \chi_{[x_0,x_1]} + \sum_{i=2}^{n} c_i \chi_{]x_{i-1},x_i]}$$

THAT IS, IN PLAIN WORDS,

IS A "PIECEWISE COSTANT FUNCTION" ON THE INTERVALS OF THE SUBDIVISION:

Φ SIMPLE MEANS:

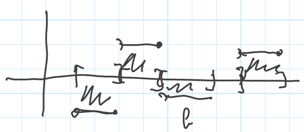


CLEARLY, IF Φ STEP FUNCT $\Rightarrow$ Φ SIMPLE.

NOW, GIVEN $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ LIMITED

LOWER $\quad R\underline{\int_a^b} f \overset{DEF}{=} \sup_{\substack{\Phi \le f \\ \Phi \text{ STEP}}} \int_a^b \Phi$

AND $\quad \int_a^b \Phi = c_1 (x_1 - x_0) + \sum_{i=2}^{n} c_i (x_i - x_{i-1}) \leftarrow \Phi$



UPPER $\quad R\overline{\int_a^b} f \overset{DEF}{=} \inf \int_a^b \Psi$

$$\int_a \; \ldots \; \inf_{\substack{\varphi \geq f \\ \varphi \text{ STEP}}} \int_a \ldots$$

DEF f IS INTEGRABLE IN THE RIEMANN SENSE (R-INTEGRABLE) $\overset{\text{DEF}}{\iff}$

$$R\underline{\int_a^b} f = R\overline{\int_a^b} f$$

THM (LEBESGUE/VITALI)

LET $f : [a,b] \longrightarrow \mathbb{R}$ LIMITED.

THEN f IS R-INTEGRABLE

$$\Updownarrow$$

f IS CONTINUOUS A.E. ON $[a,b]$.

RECALL LET $f : [0,1] \longrightarrow \mathbb{R}$ SUCH THAT

$$f(x) = \begin{cases} 1 & x \in [0,1] \setminus \mathbb{Q} \\ 0 & x \in [0,1] \cap \mathbb{Q} \end{cases}$$

DIRICHLET FUNCT

NOTICE THAT , f IS DISCONTINUOUS AT ANY POINT $x \in [0,1]$

$$\Downarrow$$

$$\{ x \in [0,1] \, ; \, f \text{ DISCONTINUOUS AT } x \} = [0,1]$$

AND HENCE IT HAS MEASURE $1 \neq 0$

$$\Downarrow \text{ L.V. THM}$$

f NOT R-INTEGRABLE ON $[0,1]$!!!

ON THE OTHER HAND , WE HAVE :

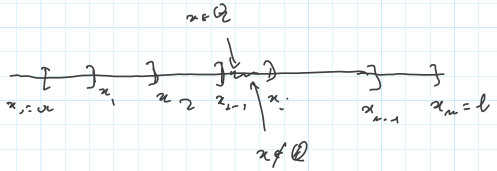
GIVEN A SUBDIVISION

$$x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \, ,$$

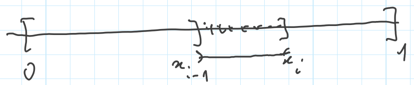
EVERY INTERVAL IN THE SUBDIVISION

EVERY INTERVAL IN THE SUBDIVISION

CONTAINS BOTH RATIONAL AND IRRATIONAL POINTS !!!



THEN $\underline{\phi}$ IS STEP FUNCT + $\underline{\phi} \leq f$



RATIONAL POINTS FOR WHICH $f(x) = 0$

$$\left\{ \begin{array}{l} \underline{\phi} \text{ STEP} \\ \underline{\phi} \leq f \end{array} \right. \qquad \underline{\phi}(x) \leq 0$$

$$\Downarrow$$

$$\underline{\phi} \leq 0 \implies R\underline{\int_a^b} f \overset{\text{DEF}}{=} \sup_{\substack{\underline{\phi} \leq f \\ \underline{\phi} \text{ STEP}}} \int_a^b \underline{\phi} \leq 0$$

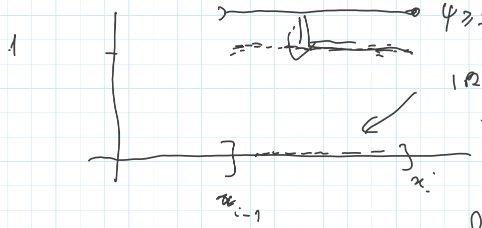
$$\sup_{\substack{\underline{\phi} \leq f \\ \underline{\phi} \text{ STEP}}} \int_a^b \underline{\phi} \geq 0$$

INDEED IS $= 0$!!!

FOR THE SAME REASON, ANY INTERVAL

$]x_{i-1}, x_i]$ IN THE SUBDIVISION

CONTAINS ALSO IRRATIONAL POINTS !!!



IRRATIONAL POINTS FOR WHICH $f(x) = 1$

HENCE,

$\bar{\psi}$ STEP FUNCT + $\bar{\psi} \geq f \implies$

$\implies \bar{\psi} \geq 1$

$$\Downarrow$$

$$R\overline{\int^b} f = \inf \int \bar{\psi} \geq 1$$

$$\int_a \quad \psi \geq f \quad \geq 1 \qquad \psi \text{ STEP}$$

ACTUALLY EQUALS 1

WE HAVE :

THM LET $f : [a,b] \longrightarrow \mathbb{R}$ LIMITED.

i) IF f IS R-INTEGRABLE OVER $[a,b]$ THEN f IS L-INTEGRABLE OVER $[a,b]$

ii) FURTHERMORE :

$$R\int_a^b f = \int_{[a,b]} f$$

RIEMANN INTEGRAL — LEBESGUE INTEGRAL

PROOF

LOWER INTEGRALS — WE RECALL :

$$R\underline{\int_a^b} f \overset{DEF}{=} \sup_{\substack{\psi \leq f \\ \psi \text{ STEP}}} \int_a^b \psi \qquad \text{(RIEMANN)}$$

$$\wedge|$$

$$\underline{\int_{[a,b]}} f \overset{DEF}{=} \sup_{\substack{\psi \leq f \\ \psi \text{ SIMPLE}}} \int \psi \qquad \text{(LEBESGUE)}$$

NOW, RECALL ψ STEP $\Longrightarrow$ ψ SIMPLE ($\not\Longleftarrow$)

THEN THE SET

$$\left\{ \int \psi \; ; \psi \text{ STEP}, \psi \leq f \right\} \subseteq \left\{ \int \psi \; ; \psi \text{ SIMPLE}, \psi \leq f \right\}$$

$$\Downarrow$$

$$\sup \leq \sup$$

HENCE, WE PROVED THAT

$$\int^b \ldots$$

$$R\underline{\int_a^b} f \le \underline{\int}_{[a,b]} f .$$

FOR THE SAME REASON:

UPPER

$$R\overline{\int_a^b} f \overset{DEF}{=} \inf_{\substack{\overline{\phi} \ge f \\ \overline{\phi} \text{ STEP}}} \int \overline{\phi} \qquad \text{RIEMANN}$$

$$\vee\!\shortmid \qquad\qquad\qquad\qquad \wedge\!\shortmid$$

$$\overline{\int}_{[a,b]} f \overset{DEF}{=} \inf_{\substack{\varphi \ge f \\ \varphi \text{ SIMPLE}}} \int \varphi \qquad \text{LEBESGUE}$$

SO WE HAVE

$$R\underline{\int_a^b} f \underset{!}{\overset{=}{\le}} \underline{\int}_{[a,b]} f \underset{!}{\overset{=}{\le}} \overline{\int}_{[a,b]} f \underset{!}{\overset{=}{\le}} R\overline{\int_a^b} f$$

LOWER LEBESGUE — UPPER LEBESGUE

EQUAL!!!

NOW, IF f IS R-INTEGRABLE $\Rightarrow$ $R\underline{\int_a^b} f = R\overline{\int_a^b} f$

$\Rightarrow$ f L-INTEGRABLE +

$$R\int_a^b f = \int_{[a,b]} f$$

RIEMANN — LEBESGUE

BREAK QUESTIONS?

BEGIN AGAIN AT 12.10