WE ARE IN STEP 2

THAT IS:

$f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ limited, $\mu(E) < +\infty$

f MEASURABLE.

THM (DOMINATED POINTWISE CONVERGENCE THM)

LET $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$ $\quad \mu(E) < +\infty$

f_n MEASURABLE AND LIMITED $\forall n \in \mathbb{N}$.

ASSUME

HP 1 $\quad f_n \xrightarrow[n \to \infty]{P.W.} f \quad$ (JUST WEAK COND: P.W. CONV !!)

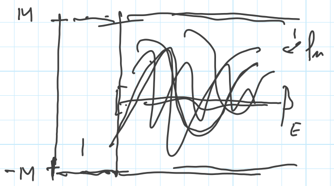
HP 2 (DOMINANCE COND) $\quad (*)$

$\exists M \in \mathbb{R}^+$ SUCH THAT

$$|f_n(x)| \leq M \quad \forall x \in E \ \forall n \in \mathbb{N}.$$

!!!

IN PLAIN WORDS:



$, \mu(E) < +\infty$

SO $(*)$ DOMINANCE $\Rightarrow$ f_n LIMITED
$\not\Leftarrow$

THEN (TH) $\quad$ $$\int_E f = \lim_{n \to \infty} \int_E f_n$$

PROOF $\quad$ WE RECALL $f_n \xrightarrow[n \to \infty]{PW} f$ AND $\mu(E) < +\infty$

f_n MEASURABLE $\forall n$

SO LITTLEWOOD LEMMA HOLDS:

THEN $\forall \varepsilon' \in \mathbb{R}^+ \quad \forall \eta' \in \mathbb{R}^+$

THERE EXISTS $B \subset E \quad \mu(B) < \eta'$

THERE EXISTS $B \subseteq E$, $\mu(B) < \eta'$

(*) AND $n_{\varepsilon', \eta'} \in \mathbb{N}$ SUCH THAT

$$|f_n(x) - f(x)| < \varepsilon' \quad \forall x \in E \setminus B \quad \forall n \geq n_{\varepsilon', \eta'}.$$

NOW, FOR EVERY $\varepsilon \in \mathbb{R}^+$ TAKE

$$\varepsilon' = \frac{\varepsilon}{2\mu(E)}, \quad \eta' = \frac{\varepsilon}{4M}$$

← M IS THE DOMINANCE CONST.

$|f_n(x)| \leq M$ $\forall n \; \forall x \in E$

LITTLEWOOD LEMMA SAYS THAT

$\exists B \subseteq E$, $\mu(B) < \frac{\varepsilon}{4M} = \eta'$ AND n_ε SUCH

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2\mu(E)} = \varepsilon' \quad \forall x \in E \setminus B$$

$$\forall n > n_\varepsilon.$$

NOW

$$(\dagger) \quad \left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - f) \right| \leq$$

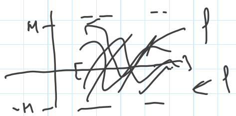
$$\underset{\text{MONOTONICITY}}{\leq} \int_E |f_n - f| \overset{\text{ADDITIVITY}}{=} \int_{E \setminus B} |f_n - f| + \int_B |f_n - f|$$

NOW 1) $\displaystyle\int_{E \setminus B} \underbrace{|f_n - f|}_{\leq \varepsilon / 2\mu(E)} \leq \frac{\varepsilon}{2\mu(E)} \cdot \mu(E \setminus B) \leq$ FOR $n > n_\varepsilon$

$$\leq \frac{\varepsilon}{2\mu(E)} \cdot \mu(E) = \varepsilon/2$$

2) $\displaystyle\int_B \underbrace{|f_n - f|}_{\leq 2M} \leq 2M \cdot \mu(B) < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}$ $\quad \mu(B) < \frac{\varepsilon}{4M}$.

DOMINANCE COND $\Rightarrow$



HENCE, FOR LARGE $n > n_\varepsilon$ WE PROVED

(†) $\left|\int_E f_n - \int_E f\right| \le$

$$\le \underbrace{\int_{E\setminus B} |f_n - f|}_{<\varepsilon/2} + \underbrace{\int_B |f_n - f|}_{<\varepsilon/2} < \varepsilon$$

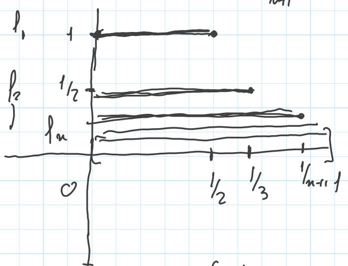
HENCE

$$\int_E f_n \xrightarrow[n\to\infty]{} \int_E f$$

<u>QED</u>

<u>EXAMPLE</u> LET $f_n : [0,1[\to \mathbb{R}$ WHERE

$$f_n = \frac{1}{n}\cdot \chi_{[0,1-\frac{1}{n+1}]} \qquad \forall n \in \mathbb{Z}^+$$



NOW

$$\int_{[0,1[} f_n = \int_{[0,1[} \frac{1}{n}\cdot \chi_{[0,1-\frac{1}{n+1}[} =$$

$$= \frac{1}{n}\cdot \underbrace{\mu\left(\left[0, 1-\frac{1}{n+1}\right[\right)}_{<1} < \frac{1}{n}$$

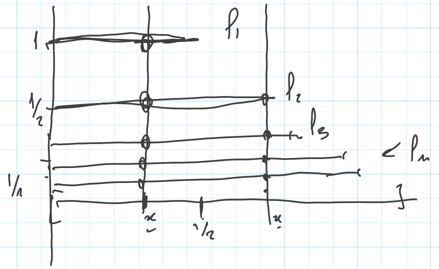
$$0 \le \int_{[0,1[} f_n \le \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

HENCE

$$\int_{[0,1[} f_n \xrightarrow[n\to\infty]{} 0$$

<u>FIRST FACT</u>

BUT, AGAIN THE SEQUENCE IS IN PLAIN WORDS



BUT CLEARLY, FOR EVERY $x \in [0,1]$,

WE HAVE

$$f_n(x) \xrightarrow[n\to\infty]{} 0 \quad \forall x \in [0,1]$$

THIS MEANS, $f_n \xrightarrow[n\to\infty]{p.w.} f \equiv 0$.

WELL, AS MATTER OF FACTS:

1) $\displaystyle\int_{[0,1]} f = 0$

$$\Rightarrow \int_{[0,1]} f = \lim_{n\to\infty} \int_{[0,1]} f_n$$

2) $\displaystyle\int_{[0,1]} f_n \xrightarrow[n\to\infty]{} 0$

NOTICE THAT OUR SEQUENCE

$$f_n = \frac{1}{n} \cdot \chi_{[0,1-\frac{1}{n+1}[}$$

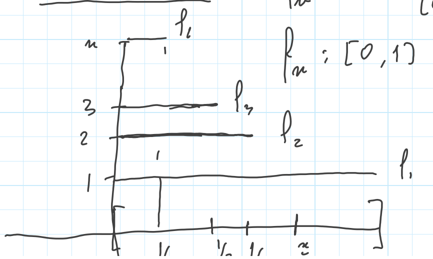
IS <u>DOMINATED</u>

FOR ISTANCE

<u>$M \geq 1$</u>

<u>COUNTER EXAMPLE</u> $f_n : n \cdot \chi_{[0,1/n[}$ $\quad n \in \mathbb{Z}^+$

$f_n : [0,1] \to \mathbb{R}$



IS $(f_n)_{n\in\mathbb{N}}$ <u>DOMINATED</u>???

$0 \mid \quad \frac{1}{n} \quad \frac{1}{3} \quad \frac{1}{2} \quad 1$ NO

FURTHERMORE $\quad f_n \xrightarrow[n\to\infty]{p.w.} f \equiv 0$

BUT

$$\int_{[0,1]} f_n = \int_{[0,1]} n\,\chi_{[0,1/n]} = n\cdot\frac{1}{n} = 1 \xrightarrow[n\to\infty]{} 1 \neq 0$$

WHILE

$$\int_{[0,1]} \underset{\overset{\shortparallel}{0}}{f} = 0$$

BREAK QUESTIONS

BYE BYE

GOOD WEEKEND