Monday, February 13, 2023 10:14 AM

STEP 3

FUNCTS $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE

AND

PLUS: f NONNEGATIVE (THAT IS: $f(x) \geq 0 \ \forall x \in E$)

RMK OBVIOUSLY, GIVEN $h : E \subseteq \mathbb{R}^n \to \mathbb{R}$, RECALL

$$\operatorname{supp}(h) = \{ x \in E ;\ h(x) \neq 0 \}.$$

THEN $\displaystyle\int_E h = \int_{\operatorname{supp}(h)} h$ (OBVIOUS).

DEFINITION $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE + NN, WE HAVE:

$$\int_E f \overset{\text{DEF}}{=} \sup_{h \leq f} \int_{\operatorname{supp}(h)} h \qquad \text{AND}$$

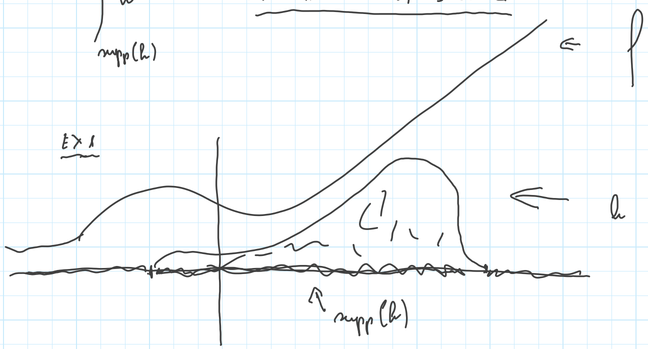
$$(*) \begin{cases} \text{i)} \ h \text{ MEASURABLE} \\ \text{ii)} \ h \text{ LIMITED} \\ \text{iii)} \ \mu(\operatorname{supp}(h)) < +\infty \end{cases}$$

SINCE $(*)$, THE h's ARE OF STEP 2 TYPE,

AND THEREFORE

$\displaystyle\int_{\operatorname{supp}(h)} h$ IS DEFINED BY STEP 2 !!!

EX 1



$E = \mathbb{R}$

EX 2 $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$

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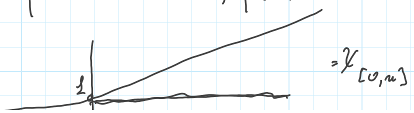
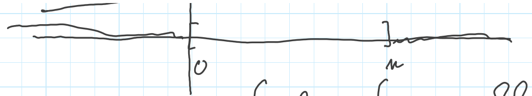




WHAT IS $\int_{\mathbb{R}} f = \int_{\mathbb{R}} e^{x}$???

NOW, FOR ANY $n \in \mathbb{Z}^+$, CONSIDER $\chi_{[0,n]}$.

THEN $\chi_{[0,n]} \leq e^{x} = f$

AND $\chi_{[0,n]}$ IS OF TYPE $(*)$.

$$+\infty = \sup_{n\in\mathbb{Z}^+} n = \sup_{n\in\mathbb{Z}^+} \int_{\mathbb{R}} \chi_{[0,n]} \leq \sup_{\substack{h\leq f \\ h \text{ OF TYPE } (*)}} \int_{\operatorname{supp}(h)} h \overset{\text{DEF}}{=} \int_{\mathbb{R}} e^{x}$$

THEN $\int_{\mathbb{R}} f = \int_{\mathbb{R}} e^{x} = +\infty$.

THM (<u>FATOU LEMMA</u> (?))

LET $(f_n)_{n\in\mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$

i) f_n MEAS + ii) f_n NONNEGATIVE $\forall n \in \mathbb{N}$.

IF $f_n \xrightarrow[n\to\infty]{\text{P.W.}} f$

THEN

$$\int_E f \leq \min\lim_{n} \int_E f_n \qquad (\dagger)$$

WHERE

$$\min\lim_{n} \int_F f_n \overset{\text{DEF}}{=} \inf_n \left(\sup_{k\geq n} \int_F f_k \right) \quad (\text{RECALL}).$$

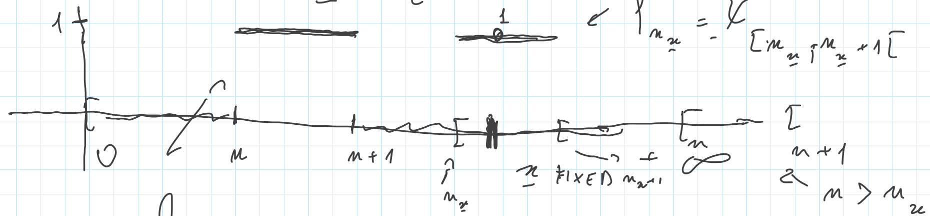
RMK THE STATEMENT OF FATOU LEMMA IS BEST POSSIBLE !!!

EX 1 LET $f_n : [0, +\infty[\to \mathbb{R}$, $f_n = \chi_{[n,n+1[}$ $n \in \mathbb{Z}^+$.

NOW

$$f_n \overset{\text{DEF}}{=} \chi_{[n,n+1[} \xrightarrow[n \to \infty]{\text{PW}} f \equiv 0 \quad (\text{WHY?})$$

FIX AN ("ARBITRARY") $\underline{x} \in [0, +\infty[$. $f_{n_{\underline{x}}} = \chi_{[n_{\underline{x}}, n_{\underline{x}}+1[}$



$$f_n = \chi_{[n,n+1]}$$

NOW $\exists !$ $n_{\underline{x}} \in \mathbb{Z}^+$ SUCH THAT $\underline{x} \in [n_{\underline{x}}, n_{\underline{x}}+1[$

THEN

$$(f_n(\underset{\text{FIXED}}{\underline{x}}))_{n \in \mathbb{N}} = (\chi_{[n,n+1[}(\underset{\text{FIXE}}{\underline{x}}))_{n \in \mathbb{N}} \overset{\text{IS}}{=}$$

$$\overset{\text{IS}}{=} (0, 0, \ldots, 0, \underset{n_x \text{ POSITION}}{1}, 0, 0, \ldots 0, \ldots) \xrightarrow[n \to \infty]{} 0$$

HENCE, FOR ANY $\underline{x} \in [0, +\infty[$, WE HAVE

$$f_n(\underline{x}) = \chi_{[n,n+1[}(\underline{x}) \xrightarrow[n \to \infty]{} 0 \quad !!!$$

THEREFORE,

$$(f_n)_{n\in\mathbb{N}} = \left(\chi_{[n,n+1[}\right)_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{PW} f \equiv 0.$$

BUT, NOW:

$$\int_{[0,+\infty[} f = 0 \quad !!!$$

BUT, $\displaystyle \min\lim_{n} \int_{[0,+\infty[} f_n \overset{DEF}{=}$

$$\overset{DEF}{=} \sup_{n} \left(\inf_{k\geq n} \int_{[0,+\infty[} f_k \right)$$

BUT, WHAT IS

$$\left(\int_{[0,+\infty[} f_n\right)_{n\in\mathbb{N}} = \left(\int_{[0,+\infty[} \chi_{[n,n+1[}\right)_{n\in\mathbb{N}} =$$

$$= (1, 1, 1, \ldots, \underset{n}{1}, \ldots, 1, \ldots)$$

NOW, GIVEN $n \in \mathbb{N}$

$$\inf_{k\geq n} \int_{[0,+\infty[} \chi_{[k,k+1[} = 1 \quad \forall n$$

$$\min\lim_{n} \int_{[0,+\infty[} f_n =$$

$$\overset{\text{DEF}}{=} \sup_n \left(\inf_{k \geq n} \int_{[0,+\infty)} f_k \right) = \sup_n (1, 1, \dots, 1, \dots) = \underline{1}$$

HENCE :

$$0 = \int_{[0,+\infty)} f \underset{\neq}{<} \min_n \int_{[0,+\infty)} f_n = 1$$

$$f \equiv 0$$

<u>BREAK</u> QUESTIONS?

<u>BEGIN AGAIN AT 12.10</u>