Monday, February 13, 2023 10:15 AM

RMK 2 $\min\lim_n \int_E f_n$ CANNOT BE IMPROVED!

IN PLAIN WORDS :

$$f_n \xrightarrow[n\to\infty]{PW} f \not\Rightarrow \left(\int_E f_n\right)_{n\in\mathbb{N}} \text{ IS CONVERGENT!}$$

FOR INSTANCE : $f_n : [0,+\infty[\to \mathbb{R}$ WHERE

$$f_n = \begin{cases} \chi_{[n,n+1[} & n \text{ EVEN} \\ 0 & n \text{ ODD} \end{cases}$$

THEN , $f_n \xrightarrow[n\to\infty]{PW} f \equiv 0 \Rightarrow \int_{[0,+\infty} f = 0$

BUT

$$\int_{[0,+\infty} f_n = (1,0,1,0,----)$$

IT IS AN OSCILLATING SEQUENCE $\Rightarrow$ NOT CONVERGENT !!!

BUT NOW

$$\min\lim_n \left(\int_{[0,+\infty[} f_n\right) \stackrel{DEF}{=}$$

$$\sup_n \left(\inf_{k\geq n} \int_{[0,+\infty[} f_k\right) =$$

$$= \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} \left(1,0,\underbrace{1,0,\ldots,1,0}_{k\geq n}\right)\right)$$

$$= \sup_n (0,0,0,---,0,--) = 0$$

THEN

$$\to \min\lim_n \int_{[0,+\infty} f_n = 0 \underset{\text{TRUE}}{\geq} 0 = \int_{[0,+\infty[} f$$

RMK 3 THE TRUE HYPOTHESIS OF FATOU LEMMA

IS f_n IS NONNEGATIVE. THE SIMPLER FACT

THAT THE LIMIT FUNCT f IS NONNEGATIVE,

IS NOT SUFFICIENT.

EX $(f_n)_{n \in \mathbb{N}}$, $f_n : [0, +\infty[\longrightarrow \mathbb{R}$

WHERE

$$f_n = \begin{cases} \chi_{[n, n+1[} & n \text{ EVEN} \\ -\chi_{[n, n+1[} & n \text{ ODD.} \end{cases}$$

SO f_n NN IS FALSE FOR ANY n ODD.

BUT $f_n \xrightarrow[n \to \infty]{PW} f \equiv 0$ WHICH IS NONNEGATIVE.

IS IT TRUE

$$\int_{[0,+\infty[} f \leq \min\lim_n \int_{[0,+\infty[} f_n \quad ???$$

WE HAVE $\int_{[0,+\infty[} f = 0$

BUT $\int_{[0,+\infty[} f_n = (1, -1, 1, -1, \dots)$

THEN

$$\min\lim_n \int_{[0,+\infty[} f_n \overset{DEF}{=}$$

$$\sup_n \left(\inf_{k \geq n} \int_{[0,+\infty[} f_k \right) = \sup_n (-1, -1, -1, \dots) = -1$$

SO

$$\min\lim_n \int_{[0,+\infty[} f_n = -1 \not\geq 0 = \int_{[0,+\infty[} f$$

TRUE

PROOF WE HAVE $f_n \xrightarrow[n\to\infty]{PW} f$ (HP)

AND f_n MEAS $\forall n$.

LET US CONSIDER A FUNCTION $h : E \subseteq \mathbb{R}^n \to \mathbb{R}$

SUCH THAT

$$(**)\left\{\begin{array}{l}(*)\left\{\begin{array}{l} i)\ h \text{ MEAS} \\ ii)\ h \text{ LIMITED} \\ iii)\ \mu(\operatorname{supp}(h)) < +\infty \end{array}\right. \\ \quad + \\ h \le f \end{array}\right.$$

THIS LEADS TO

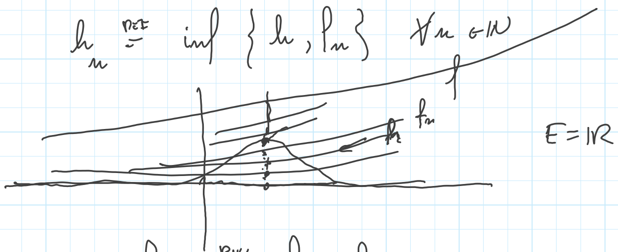
$$\int_E f \overset{\text{DEF}}{=} \sup_{\substack{h \le f \\ h \text{ OF TYPE } (*)}} \int_{\operatorname{supp}(h)} h$$

MAIN DEFINITION

LET US FIX A FUNCTION h OF TYPE $(**)$.

CONSIDER THE FUNCTS:

$$h_n \overset{\text{DEF}}{=} \inf\{h, f_n\} \quad \forall n \in \mathbb{N}$$



$E = \mathbb{R}$

SINCE $f_n \xrightarrow[n\to\infty]{PW} f \ge h$ IT FOLLOWS THAT

$$h_n = \inf(f_n, h) \xrightarrow[n\to\infty]{PW} h \quad !!!$$

FIRST FACT

BUT, SINCE h LIMITED FUNCT, THEN

$$\exists M \in \mathbb{R}^+ \text{ WHERE } M > \sup\{h(x); x \in E\} < +\infty$$

$$\Rightarrow \quad |h_n(x)| < M \quad \forall x \in E \ \forall n \in \mathbb{N}$$

$\Rightarrow (\ x \in \mathrm{supp}(h)\ )$

DOMINANCE CONDITION

WE ARE IN THE CONDITIONS OF

THE DOMINATED PW CONVERGENCE THM OF STEP 2) $\Rightarrow$

$$\underline{\underline{\int_{\mathrm{supp}(h)} h}} \overset{!}{=} \lim_{n\to\infty} \int_{\mathrm{supp}(h)} h_n =$$

$$= \min\lim_n \int_{\mathrm{supp}(h)} h_n \leq \underline{\underline{\min\lim_n \int f_n}}$$

NOW, SINCE $h_n \leq f_n$ $\forall n \in \mathbb{N} \Rightarrow \int h_n \leq \int f_n$ $\forall n \in \mathbb{N}$

SO WE GET

$$\underline{\int_{\mathrm{supp}(h)} h \leq \min\lim_n \int_E f_n}$$

IT IS TRUE FOR ANY FUNCT h OF TYPE $(*)$

THAT IS
i) h MEAS
ii) h LIMITED
iii) $\mu(\mathrm{supp}(h)) < +\infty$
$+$
$h \leq f$

HENCE

$$\int_E f \overset{\mathrm{DEF}}{=} \sup_{\substack{h \leq f \\ h \text{ OF} \\ \text{TYPE}(*)}} \int_{\mathrm{supp}(h)} h \leq \min\lim_n \int_E f_n \quad \underline{\text{QED}}$$

BREAK QUESTIONS?

BYE BYE