Monday, February 13, 2023 10:16 AM

STEP 3) FUNCTIONS

$$f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}} \quad \text{MEASURABLE N/N.}$$

RECALL.

$$\int_E f = \sup_{h \leq f} \int_{\operatorname{supp}(h)} h$$

WHERE: i) h MEASURABLE

ii) h LIMITED

iii) $\mu(\operatorname{supp}(h)) < +\infty$.

RECALL

THM (FATOU LEMMA) $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$

f_n MEAS + NN.

IF $f_n \xrightarrow[n \to \infty]{\text{PW}} f$ THEN

$$\int_E f \leq \liminf_n \int_E f_n \quad . \boxtimes$$

THE BEPPO LEVI THMS.

THM (FIRST FORM / MONOTONE CONV. THM)

$(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE N.N.

SUPPOSE THAT $(f_n)_{n \in \mathbb{N}}$ MONOTONE NON DECREASING,

THAT

$$f_n \leq f_{n+1} \quad \forall n \in \mathbb{N}.$$

THEN, BY SETTING (?)

(γ) $$f = \lim_{n \to \infty} f_n \quad \text{(POINTWISE LIMIT)} \quad (?)$$

POSITION

NOT

WE HAVE

BY HYPOTHESIS WE HAVE

$$\int_E f = \lim_{n\to\infty} \int_E f_n \;.\; \square$$

CLEAR RMK (*) :

IF $f_n \le f_{n+1}$ $n \in \mathbb{N}$ (MONOTONICITY)

THEN $f_n \xrightarrow[n\to\infty]{PW} f \overset{!}{=} \sup_n f_n$!!!

GIVEN A FIXED $\underline{x} \in E$, STUDY THE SEQUENCE

OF EVALUATIONS :

$$\left(f_n(\underline{x})\right)_{n\in\mathbb{N}} \quad !!!$$

NOW

$$\min\lim_n f_n(\underline{x}) = \sup_n \left(\inf_{k\ge n} f_k(\underline{x})\right) :$$

BUT GIVEN $n \in \mathbb{N}$ WHAT IS

$$\left(f_k(\underline{x})\right)_{k\ge n} = \left(f_n(\underline{x}) \le f_{n+1}(x) \le \ldots\right)$$

$$\Rightarrow \inf_{k\ge n} f_k(\underline{x}) = f_n(x) \Rightarrow$$

$$\min\lim_n f_n(\underline{x}) = \sup_n \left(\inf_{k\ge n} f_k(\underline{x})\right) = \sup_n \left(f_0(\underline{x}), f_1(x) \ldots f_n(x)\right)$$

$$= \sup_n f_n(\underline{x}) \qquad (\dagger)$$

NOW $\max\lim_n f_n(\underline{x}) \overset{\text{DEF}}{=} \inf_n \left(\sup_{k\ge n} f_k(\underline{x})\right) = \sup_n f_n(\underline{x})$

↑ ALL EQUAL TO $\sup_n f_n(\underline{x})$

TO SUMMARIZE, $\forall \underline{x} \in E$ WE HAVE

$$\min\lim_n f_n(\underline{x}) = \sup_n f_n(\underline{x}) = \max\lim_n f_n(\underline{x})$$

THEREFORE

$$f_n \xrightarrow[n\to\infty]{pw} \sup_n f_n(x) = \underline{\lim_{n\to\infty} f_n(x)}.$$

THIS IS TRUE FOR EVERY $\underline{x} \in E \Longrightarrow$

$$\boxed{f_n \xrightarrow[n\to\infty]{pw} f = \sup_n f_n .}$$

<u>PROOF</u> NOTICE THAT

$$\lim_{n\to\infty} f_n = f = \sup_n f_n \Longrightarrow \underline{f_n \leq f} \quad \text{OBVIOUS} \overset{\text{MONOTONICITY}}{\Longrightarrow}$$

$$\Longrightarrow \int_E f_n \leq \int_E f \Longrightarrow$$

$$\max_n \lim \int_E f \leq \int_E f \quad . \quad (1).$$

BUT, RECALL $f_n \xrightarrow[n\to\infty]{pw} f \overset{\text{FATOU LEMMA}}{\Longrightarrow}$

$$\Longrightarrow \int_E f \leq \min_n \lim \int_E f_n \quad (2)$$

HENCE (1) & (2) IMPLY:

$$\max_n \lim \int_E f_n \overset{(1)}{\leq} \int_E f \overset{(2)}{\leq} \min_n \lim \int_E f_n \quad (\dagger) \leftarrow$$

BUT, IN GENERAL

$$\min_n \lim \leq \max_n \lim .$$

HENCE

$$\min_n \lim \int_E f_n = \max_n \lim \int_E f_n \overset{\text{DEF}}{=} \lim_{n\to\infty} \int_E f_n \overset{!}{=} \int_E f \quad . \underline{\text{QED.}}$$

<u>THM</u> (<u>BEPPO LEVI SECOND VERSION</u>)

(f_n), $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ <u>MEASURABLE FUN</u>.

MAKE HYPOTHESIS: $\left\{\begin{array}{l} i)\ f_n \xrightarrow[n\to\infty]{P.W.} f \\ ii)\ f_n \le f \quad \forall n \in \mathbb{N}. \end{array}\right.$ $(\#)$

THEN

$$\int_E f = \lim_{n\to\infty} \int_E f_n . \qquad \boxed{}$$

THE PROOF IS THE SAME!!!

TYPICAL / IMPORTANT APPLICATION.

1) $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE NN.

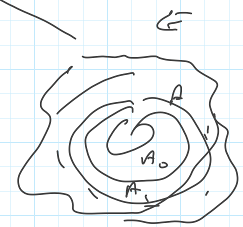
GIVEN A NESTED INCREASING SEQUENCE

$(A_n)_{n\in\mathbb{N}}$, $A_n \subseteq E$, A_n MEASURABLE

SUCH THAT

i) $A_n \subseteq A_{n+1} \quad \forall n$

ii) $\bigcup_n A_n = E$.



THEN

$$\int_{E=\bigcup_n A_n} f = \lim_{n\to\infty} \int_E f\cdot \chi_{A_n}$$

PROOF LET US CONSIDER THE SEQUENCE

$(f\cdot\chi_{A_n})_{n\to\infty} \xrightarrow[n\to\infty]{PW} ?$ IF YES, TO WHICH FUNCT ??

FIX $x \in E$ AND STUDY

FIX $\underline{x} \in E$ AND STUDY

$$\left(f \cdot \chi_{A_n}(\underline{x})\right)_{n \in \mathbb{N}} =$$

$$= \left(f \cdot \chi_{A_0}(\underline{x}), f \cdot \chi_{A_1}(\underline{x}), \dots, f \cdot \chi_{A_n}(\underline{x}) \dots \right)$$

BUT $\quad A_n \subseteq A_{n+1}, \forall \quad$ AND $\underline{\underline{\bigcup_n A_n = E}}.$

$\in$

$\exists\, n_{\underline{x}} \;;\; \underline{x} \in A_{n_{\underline{x}}} \implies$

$\forall n \geq n_x \quad \underline{x} \in A_n$

SO $\quad \left(f \cdot \chi_{A_n}(\underline{x})\right)_{n \in \mathbb{N}} =$

$$= \left(0, 0, \dots, 0, f \cdot \underbrace{\overset{=1}{\chi_{A_{n_{\underline{x}}}}}(\underline{x})}_{n_{\underline{x}}} = f(\underline{x}), f(\underline{x}), f(\underline{x}) \dots \right) \xrightarrow[n \to \infty]{} f(\underline{x}) \quad \forall \underline{x} \in E$$

THEN

$$f \cdot \chi_{A_n} \xrightarrow[n \to \infty]{pw} f \qquad (1)$$

BUT, SINCE f NON NEGATIVE AND $A_n \subseteq A_{n+1}$

$$f \cdot \chi_{A_n} \leq f \cdot \chi_{A_{n+1}} \qquad \text{MONOTONE} \quad (2)$$

FROM (1) AND (2), BY BEPPO LEVI

$$\underline{\int_E f} = \lim_{n \to \infty} \int_E f \cdot \chi_{A_n} = \lim_{n \to \infty} \underline{\int_{A_n} f}$$

BREAK $\quad$ QUESTIONS?

BEGIN AGAIN AT 15.15