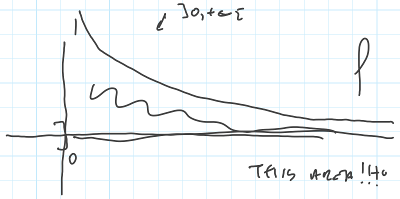
Monday, February 13, 2023 10:16 AM

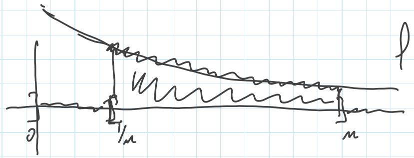
EX LET $f :]0,+\infty[\to \mathbb{R}$, $f(x) = \frac{1}{x}$, $x \in]0,+\infty[$.

COMPUTE $\displaystyle\int_{]0,+\infty[} f$!!!



FOR $n \in \mathbb{Z}^+$, LET

$$A_n = [1/n, n]$$



$f_n = f \cdot \chi_{A_n}$

$(A_n = [1/n, n])_{n \in \mathbb{Z}^+}$ IS NESTED, THAT IS

$[1/n, n] \subseteq [\frac{1}{n+1}, n+1]$ $\forall n$.

f NONNEGATIVE

THEN

$$\int_{]0,+\infty[} f \overset{DEF}{=} \int_{]0,+\infty[} \frac{1}{x} \overset{THM}{=} \lim_{n\to\infty} \int_{]0,+\infty[} f \cdot \chi_{[1/n,n]} = \lim_{n\to\infty} \int_{[1/n,n]} \frac{1}{x}$$

$f(x) = 1/x$ IS $\underline{\text{LIMITED + CONT ON}}$ $[1/n, n]$

$$\int_{[1/n,n]} \frac{1}{x} = R\int_{1/n}^{n} \frac{1}{x}\,dx =$$

$$= \Big[\log x\Big]_{x=1/n}^{x=n} =$$

$$= \log n - \log(1/n) \xrightarrow[n\to\infty]{} +\infty - (-\infty) =$$

$$= +\infty + \infty = +\infty$$

HENCE

$$\int_{]0,+\infty[} 1/x = \lim_{n\to\infty} R\int_{1/n}^{n} 1/x \, dx = \underline{+\infty} \quad (\underline{\text{DONE}}\ !!!)$$

<u>EX 2</u> LET $f : [1,+\infty[\to \mathbb{R}$, $f(x) = \frac{1}{x^2}$, $x \in [1,+\infty[$

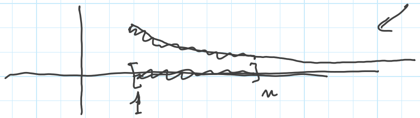
COMPUTE $\int_{[1,+\infty[} f$!!!

NOW, LET $A_n = [1, n]$, THEN

A_n IS NESTED, THAT IS $[1,n] \subseteq [1,n+1]$ $\forall$

AND $[1,+\infty[= \bigcup_n [1,n]$

IN PLAIN WORDS:



AND

$$\int_{[1,+\infty[} f = \lim_{n\to\infty} \int_{[1,n]} f =$$

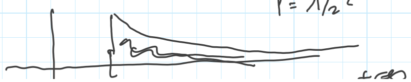
$$= \lim_{n\to\infty} R\int_1^n \frac{1}{x^2}\,dx$$

BUT FOR n FIXED

$$R\int_1^n \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{x=1}^{x=n} = -\frac{1}{n} - (-1) = -1/n + 1$$

HENCE

$$\int_{[1,+\infty[} f = \int_{[1,+\infty[} \frac{1}{x^2} = \lim_{n\to\infty} \left(-1/n + 1\right) = \underline{1}$$



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RMK REMEMBER $(A_n)_{n\in\mathbb{N}}$, A_n MEAS,

$$(*)\quad A_n \subseteq A_{n+1} \Rightarrow \mu\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \mu(A_n) \quad \text{OLD RESULT}$$

AS A MATTER OF FACT, IT CAN BE NOW REGARDED A SIMPLE SPECIAL CASE OF THE PRECEDING APPLICATION OF BEPPO LEVI THM!!!

INDEED $(*) \Rightarrow$ AND $A_n \subseteq A_{n+1}$

$$\chi_{\bigcup_n A_n} = \lim_{n\to\infty} \chi_{A_n} \qquad \Downarrow$$

$$\chi_{A_n} \leq \chi_{A_{n+1}} \quad \text{MONOTONE NON DECREASING}$$

$\Downarrow$

BY BEPPO LEVI THM $\Rightarrow$

$$\mu\left(\bigcup_n A_n\right) = \int_{\mathbb{R}^n} \chi_{\bigcup_n A_n} = \lim_{n\to\infty} \int_{\mathbb{R}^n} \chi_{A_n} = \lim_{n\to\infty} \mu(A_n)$$

$$\int_{\mathbb{R}^n} \chi_{\bigcup_n A_n} = \mu\left(\bigcup_n A_n\right), \qquad \int_{\mathbb{R}^n} \chi_{A_n} = \mu(A_n)$$

THE OLD RESULT !!!

THE BEPPO LEVI THM FOR SERIES

TO BE PRECISE:

$(f_n)_{n\in\mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEAS FUN.

CONSIDER THE ASSOCIATED SERIES:

$$(\dagger)\quad \sum_{n=0}^{\infty} f_n \overset{\text{DEF}}{=} \left(\sum_{k=0}^{n} f_k\right)_{n\in\mathbb{N}} \qquad \text{DEFINITION OF SERIES}$$

SERIES

THE SUM OF $(\dagger)$ IS (IF IT EXISTS ?!)

$$g = \sum_{n=0}^{\infty} f_n \overset{\text{DEF}}{=} \lim_{n\to\infty} \left(\sum_{k=0}^{n} f_k\right)$$

BUT NOW, UNDER OUR HYPOTHESIS

$$\sum_{k=0} f_k = \sum_{k=0} f_k + \underset{\underset{0}{\vee\!\!|}}{f_{n+1}} \geq \sum_{k=0} f_k \Rightarrow$$

$\left(\sum_{k=0}^{n} f_k\right)_{n\in\mathbb{N}}$ IS <u>MONOTONE</u> (NON DECREASING)

AND HENCE <u>PW CONVERGE</u>

THEN, BY BEPPO LEVI

$$\int_E g = \int_E \left(\sum_{n=0}^{\infty} f_n\right) = \lim_{n\to\infty} \left(\int_E \left(\sum_{k=0}^{n} f_k\right)\right) =$$

$\Updownarrow$ LINEARITY

$$= \lim_{n\to} \left(\sum_{k=0}^{n} \int_E f_k\right)$$

$$\stackrel{\text{DEF}}{=} \sum_{n=0}^{\infty} \int_E f_n$$

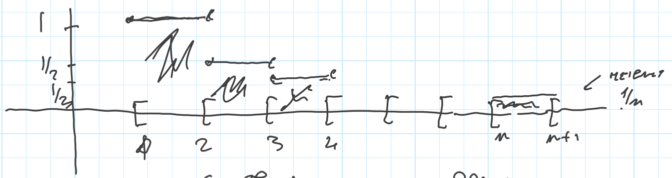
IN PLAIN WORDS: $f_n \geq 0 \ \forall n$

$$\int_E g = \int_E \left(\sum_{n=0}^{\infty} f_n\right) = \sum_{n=0}^{\infty} \int_E f_n \qquad !!!$$

<u>EX</u> LET $f_n = \frac{1}{n} \cdot \chi_{[n,n+1[} \quad \forall n \in \mathbb{Z}^+$

AND $g = \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} 1/n \cdot \chi_{[n,n+1[}$

$\downarrow$ GRAPHICALLY



WHAT IS $\int_{[1,+\infty[} \left(\sum_{n=1}^{\infty} \frac{1}{n} \chi_{[n,n+1[}\right) = ???$

$\|$ BY BEPPO LEVI FOR SERIES

$$\sum_{n=1}^{\infty} \int \frac{1}{n} \chi_{[n,n+1[} = \sum_{n=1}^{\infty} 1/n = +\infty$$

$1/n \cdot \mu([n,n+1[) = 1/n$

THE CLASSICAL HARMONIC SERIES

STOP QUESTIONS?

BYE BYE