Monday, February 13, 2023 10:17 AM

DEF LET $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE N/N.

f IS SAID TO SUMMABLE $\overset{DEF}{\iff} \int_E f < +\infty$.

EX 1 LET $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x}$, MEAS + NN

QUESTION: LET $E = [0, +\infty[$. IS f SUMMABLE OVER $E = [0, +\infty[$?

WE HAVE TO COMPUTE $\int_{[0,+\infty[} f$ (!) ...

GIVEN $n \in \mathbb{Z}^+$, CONSIDER $[0, n]$ $\forall n \in \mathbb{Z}^+$.

$$[0,n] \subseteq [0, n+1] \quad + \quad \bigcup_{n \in \mathbb{Z}^+} [0,n] = [0, +\infty[= E$$

THEN, BY BEPPO LEVI,

$$\int_{[0,+\infty[} e^{-x} = \lim_{n\to\infty} \int_{[0,+\infty[} e^{-x} \cdot \chi_{[0,n]} = \lim_{n\to\infty} \int_{[0,n]} e^{-x} .$$

$$= \lim_{n\to\infty} R \int_0^n e^{-x}\, dx =$$

$$= \lim_{n\to\infty} \left[-e^{-x} \right]_{x=0}^{x=n} =$$

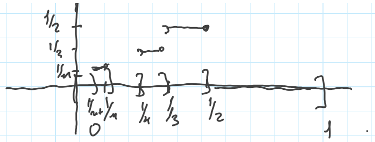
$$= \lim_{n\to\infty} \left(\underbrace{-e^{-n}}_{\downarrow\, n\to\infty \atop 0} + 1 \right) = 1 \implies$$

$$\int_{[0,+\infty[} e^{-x} = 1 < +\infty \implies e^{-x} \text{ SUMMABLE OVER } E = [0, +\infty[.$$

EX 2 CONSIDER THE SERIES

$$\to g = \sum_{n=1}^{\infty} \frac{1}{n} \underbrace{\chi_{]\frac{1}{n+1}, \frac{1}{n}]}}_{\geq 0}$$

BEPPO LEVI THM FOR SERIE APPLIES!



$$\int_{]0,1]} g = \int_{]0,1]} \left(\sum_{n=1}^{\infty} \frac{1}{n} \cdot \chi_{]\frac{1}{n+1}, \frac{1}{n}]} \right)$$

$$\overset{TBM}{=} \sum_{n=1}^{\infty} \int_{]0,1]} 1/n \cdot \chi_{]\frac{1}{n+1}, \frac{1}{n}]} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \mu \left(\underbrace{]\tfrac{1}{n+1}, \tfrac{1}{n}]}_{\frac{1}{n(n+1)}} \right) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{n(n+1)} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3 + n^2} < \sum_{n=1}^{\infty} \frac{1}{n^3} < +\infty$$

GENERALIZED HARMONIC SERIES OF EXP $3 > 1$

CONVERGENT

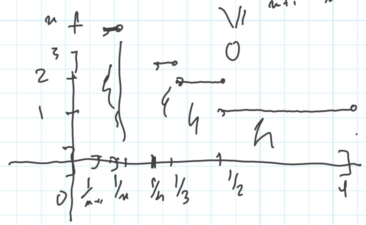
$$\int_{]0,1]} \left(\sum_{n=1}^{\infty} \frac{1}{n} \cdot \chi_{]\frac{1}{n+1}, \frac{1}{n}]} \right) < +\infty$$

THEN THE SERIES IS SUMMABLE OVER $]0,1]$.

EX 3 CONSIDER THE SERIES

$$g = \sum_{n=1}^{\infty} (n+1) \chi_{]\frac{1}{n+1}, \frac{1}{n}]} :]0,1] \longrightarrow \mathbb{R}.$$

$$\geq 0$$



IS g SUMMABLE OVER $]0,1]$?

WE HAVE TO COMPUTE:

$$\int g = \int \left(\sum^{\infty} (n+1) \cdot \chi \right)$$

BEPPO LEVI

$$\int_{]0,1]} g = \int_{]0,1]} \Big(\sum_{n=1}^{\infty} (n+1) \chi_{]\frac{1}{n+1}, \frac{1}{n}]} \Big)$$

$$= \sum_{n=1}^{\infty} \int_{]0,1]} (n+1) \chi_{]\frac{1}{n+1}, \frac{1}{n}]} =$$

$$= \sum_{n=1}^{\infty} (n+1) \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

IS DIVERGENT

HENCE $g = \sum_{n=1}^{\infty} (n+1) \chi_{]\frac{1}{n+1}, \frac{1}{n}]}$

IS NOT SUMMABLE OVER $]0,1]$. $\boxed{B}$

RECALL GENERALIZED HARMONIC SERIES, THAT IS

FOR $\alpha \in \mathbb{R}$, $\alpha \geq 1$ THE SERIES

$$(*) \quad \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

α IS THE EXPONENT

THE MAIN FACTS

1) IF $\alpha = 1 \implies (*)$ DIVERGENT (CLASSICAL HARMONIC SERIES)

$(*) = \sum_{n=1}^{\infty} \frac{1}{n}$ $\neq$ NOW $\frac{1}{n} \xrightarrow[n \to \infty]{} 0$

BUT

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty \text{ DIVERGENT.}$$

2) IF $\alpha > 1 \implies (*) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ IS CONVERGENT

THAT IS $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} < +\infty$.

STEP 4 WE CONSIDER ARBITRARY

$$f : E \subseteq \mathbb{R}^n \longrightarrow \overline{\mathbb{R}} \quad \text{MEASURABLE}$$

WITH RESPECT TO STEP 3, WE TRY TO DROP

NON NEGATIVITY CONDITION

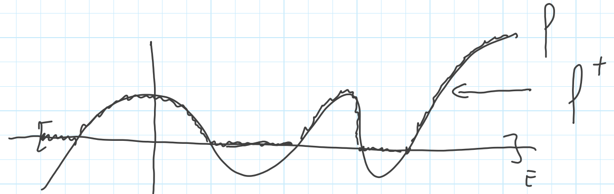
STRATEGY GIVEN $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE

WE DEFINE TWO ASSOCIATED FUNCTS:

1) POSITIVE PART OF f:

$$f^+ \quad \text{WHERE} \quad f^+(x) = \begin{cases} f(x), & \text{WHENEVER } f(x) \geq 0 \\ 0 & \text{OTHERWISE} \end{cases} \qquad (1)$$

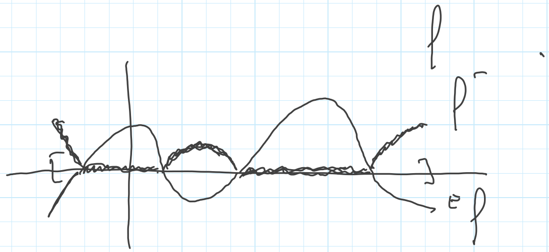
EX



2) NEGATIVE PART OF f:

$$f^- \quad \text{WHERE} \quad f^-(x) = \begin{cases} -f(x) & \text{WHENEVER } f(x) < 0 \\ 0 & \text{OTHERWISE} \end{cases} \qquad (2)$$

EX



CLEARLY, f^+ AND f^- ARE NONNEGATIVE.

BUT

$$(1) \iff f^+ = \sup\{\overset{\text{MEAS}}{f}, \overset{\text{MEAS}}{0}\} \ \text{MEAS}$$

$$(2) \iff f^- = \sup\{\underset{\text{MEAS}}{-f}, 0\} \ \text{MEAS}$$

SO, BOTH f^+ AND f^- MEAS + NN $\xrightarrow{\text{BY STEP 3}}$

THE INTEGRALS

$\int f^+$ AND $\int f^-$ ARE DEFINED (BY STEP 3)

NOW, IT IS CLEAR

i) $\underbrace{f = f^+ - f^-}$ AND ii) $|f| = f^+ + f^-$

$f = f^+ - f^-$ WE TRY TO SET

$$(\dagger) \qquad \int_E f = \int_E f^+ - \int_E f^- \qquad ???$$

DOES $(\dagger)$ MAKE SENSE IN ANY SITUATION?

NO, IN SITUATION $\int_E f^+ , \int_E f^- = +\infty$

BECAUSE $\int_E f^+ - \int_E f^- = \infty - \infty$ <u>MAKES NO SENSE</u>

IN ALL OTHER CASES $(\dagger)$ MAKES SENSE

i) $\underset{\substack{= \\ \infty}}{\int_E f^+} - \underset{<}{\int_E f^-} = +\infty$

ii) $\underset{\substack{< \\ +\infty}}{\int_E f^+} - \underset{=\infty}{\int_E f^-} = -\infty$

iii) $\underset{\substack{< \\ +\infty}}{\int_E f^+} - \underset{< +\infty}{\int_E f^-}$. IT IS A REAL NUMBER AND WE SAY

iff THAT f IS <u>SUMMABLE</u>

BREAK

QUESTIONS?

BEGIN AGAIN AT 12.15