Monday, February 13, 2023 10:18 AM

STEP 4

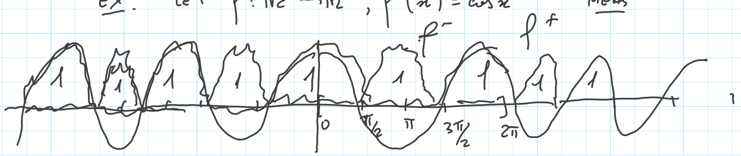
THE INTEGRAL OF $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE

IS DEFINED WITH THE EXCEPTION

IN WHICH

$$\int_E f^+ \, , \int_E f^- = +\infty \quad !!!$$

EX. LET $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$ MEAS



$$\int_{\mathbb{R}} f^+ = +\infty \qquad \int_{\mathbb{R}} f^- = +\infty$$

THEN $f = \cos x$ DOESN'T ADMIT INTEGRAL OVER $\mathbb{R}$

MAIN PROP f, g SUMMABLE OVER E.

1) $$\int_E c \cdot f = c \cdot \int_E f \qquad c \in \mathbb{R}$$

2) $$\int_E f + g = \int_E f + \int_E g \, , \quad \int_E f - g = \int_E f - \int_E g$$

3) $$f \leq g \text{ A.E.} \Rightarrow \int_E f \leq \int_E g$$ MONOTONICITY

4) LET $E = A_1 \cup A_2$ WHERE $A_1 \cap A_2 = \emptyset$

$$\int_E f = \int_{A_1} f + \int_{A_2} f$$ ADDITIVITY

THM (LEBESGUE DOMINATED CONV. THM)

LET $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE

HP1 $\quad f_n \xrightarrow[n\to\infty]{PW} f$

HP2 GENERAL DOMINANCE HYPOTHESIS

LET $\quad g : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ SUMMABLE SUCH THAT

$$|f_n| \leq g \quad \forall n \in \mathbb{N}$$

THEN i) f IS SUMMABLE.

ii) $$\int_E f = \lim_{n\to\infty} \int_E f_n .$$

PROOF i) $|f_n| < g \Rightarrow |f_n| = f_n^+ + f_n^- \leq g$

$$\Rightarrow \int_E f_n^+ , \int_E f^- < \int_E g \overset{HP}{<} +\infty$$

$$f_n \xrightarrow[n\to\infty]{PW} f \Rightarrow |f| \leq g \Rightarrow \int_E |f| = \int_E f^+ + \int_E f^- < \int_E g < +\infty$$

$\Rightarrow f$ SUMMABLE

PROOF of ii) NOTICE THAT

(*) $(g - f_n)$ IS MEAS N/N AND

$\forall n \in \mathbb{N}$

(**) $(g - f_n) \xrightarrow[n\to\infty]{PW} g - f \quad$ P.D.P.

(*) AND (**) IMPLIES THAT FATOU LEMMA HOLDS:

THEN

$$\int (g - f) \overset{\text{FATOU LEMMA}}{\leq} \min\lim \int (g - f_n) \quad (+).$$

NOW

$$\int_E g - \int_E f \overset{\text{LINEARITY}}{=} \int_E (g - f) \overset{\text{F-TOU LEMMA}}{\leq}$$

$$\leq \liminf_n \int_E (g - f_n) \overset{\text{DEF}}{=}$$

$$= \sup_n \left(\inf_{k \geq n} \left(\int_E (g - f_k) \right) \right) = \sup_n \left(\inf_{k \geq n} \left(\int_E g - \int_E f_k \right) \right)$$

$$= \sup_n \left(\int_E g - \sup_{k \geq n} \int_E f_k \right) =$$

$$= \int_E g - \inf_n \left(\sup_{k \geq n} \int_E f_k \right) =$$

$$= \int_E g - \limsup_n \int_E f_n$$. HENCE, WE PROVED:

$$\int_E g - \int_E f \leq \int_E g - \limsup_n \int_E f_n$$

$$\Downarrow$$

$$\limsup_n \int_E f_n \leq \int_E f$$ FACT 1)

NOW, PLAY THE SAME GAME: WE HAVE

(+) $(g + f_n)$ MEAS NN

FATOU LEMMA

$$(\#)\quad g+f_n \xrightarrow[n\to\infty]{PW} g+f \qquad \text{HOLDS.}$$

$$\Downarrow$$

$$\to \int_E (g+f) \le \liminf_n \int_E (g+f_n)$$

NOW

$$\underline{\int_E g + \int_E f} \overset{LIN}{=} \int_E (g+f) \overset{\substack{FATOU\\LEMMA}}{\le}$$

$$\le \liminf_n \int_E (g+f_n) \overset{DEF}{=}$$

$$\overset{DEF}{=} \sup_n \left(\inf_{k\ge n} \left(\int_E (g+f_k) \right) \right) =$$

$$= \sup_n \left(\inf_{k\ge n} \left(\int_E g + \int_E f_k \right) \right) =$$

$$= \sup_n \left(\int_E g + \inf_{k\ge n} \int f_k \right) =$$

$$= \int_E g + \sup_n \left(\inf_{k\ge n} \int f_k \right) =$$

$$= \underline{\int_E g + \liminf_n \int_E f_n}$$

WE PROVED

$$\int_E g + \int_E f \le \int_E g + \liminf_n \int_E f_n$$

$$\Downarrow$$

$$\int f \quad \le \quad \liminf \quad \int f \qquad \text{FACT 2)}$$

$$\int_E f \leq \min_n \int_E f_n$$

BUT $\quad \min h_n \leq \max h_n \quad (*)$

WHILE FACTS 1) and 2)

$$\underbrace{\max_n h_n \int_E f_n \overset{(1)}{\leq} \int_E f}_{\text{EQUAL}} \overset{(2)}{\leq} \min_n h_n \int_E f_n$$

HENCE

$$\min_n h_n \int_E f_n = \max_n h_n \int_E f_n \overset{!}{=} \lim_{n\to\infty} \int_E f_n = \int_E f$$

QED !!!

STOP QUESTIONS ?

BYEBYE GOODWEEKEND