Monday, February 13, 2023 10:18 AM

RMK

LEBESGUE DOMINATED CONV THM (STEP 4)

$\Downarrow$

DOMINATED PW CONV THM (STEP 2) :

RECALL THM LET $E \subseteq \mathbb{R}^n$, $\mu(E) < +\infty$.

LET $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$ MEASURABLE

SUCH THAT

i) $f_n \xrightarrow[n \to \infty]{PW} f$

ii) (DOMINANCE) $\exists$ $M \in \mathbb{R}^+$ SUCH THAT

(*) $|f_n(x)| < M \quad \forall n \in \mathbb{N} \ \forall x \in E$.

THEN

$$\int_E f = \lim_{n \to \infty} \int_E f_n \qquad (\text{"OLD" ONE})$$

NEW PROOF LET $g : E \to \mathbb{R}$ S.T.

$g(x) = M \quad \forall x \in E$.

(*) $\Leftrightarrow$ $|f_n| < g$ +

g SUMMABLE. INDEED $\int_E g = M \cdot \underset{+\infty}{\underset{\wedge}{\mu(E)}} < +\infty$

FROM THIS CONDS + LEBESGUE DOM. CONV THM

$$\int_E f = \lim_{n \to \infty} \int_E f_n .$$

FINAL THEMES OF OUR COURSE.

EXPLICIT COMPUTATION OF

i) $\mu(A)$, $A \subseteq \mathbb{R}^n$, A MEASURABLE.

ii) $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE

COMPUTE $\int_E f$!!!

<u>CASE i)</u> <u>TONELLI THEOREM</u>

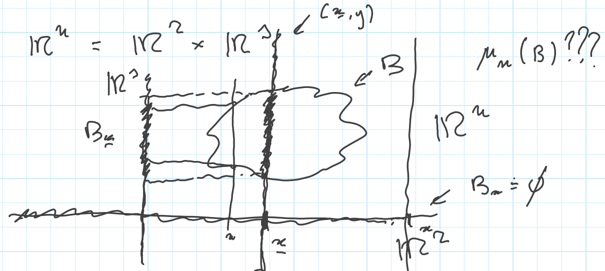
<u>PROBLEM</u> : GIVEN $B \subseteq \mathbb{R}^n$, B MEASURABLE $n \in \mathbb{Z}^+$

COMPUTE μ_n ???

GIVEN $r, s \in \mathbb{Z}^+$ SUCH THAT $r + s = n$

WE SPLIT $\mathbb{R}^n$ IN THE FORM:

$$\mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^s$$



1) LET, FOR $x \in \mathbb{R}^r$,

$$B_x \overset{\text{DEF}}{=} \{ y \in \mathbb{R}^s \ ; \ (x, y) \in B \}$$

THE <u>SECTION OF B WITH RESPECT TO $x \in \mathbb{R}^r$</u>

2) <u>SUPPORT OF B</u> :

$$S_B \overset{\text{DEF}}{=} \{ x \in \mathbb{R}^r \ ; \ \underline{\mu_s^*(B_x) > 0} \} \subseteq \mathbb{R}^r$$

$\underline{B_x \subseteq \mathbb{R}^s}$!!

3) <u>"BAD SUPPORT"</u> :

$$S_B^0 = \{ x \in \mathbb{R}^r \ ; \ B_x \text{ IS NOT MEASURABLE} \} \subseteq \mathbb{R}^r$$

CLEARLY $S_B^0 \subseteq S_B$

<u>THEN</u> :

i) S_B, S_B^0 ARE MEASURABLE SETS IN $\mathbb{R}^r$.

ii) $\mu_2(S_B) = 0$

THEN

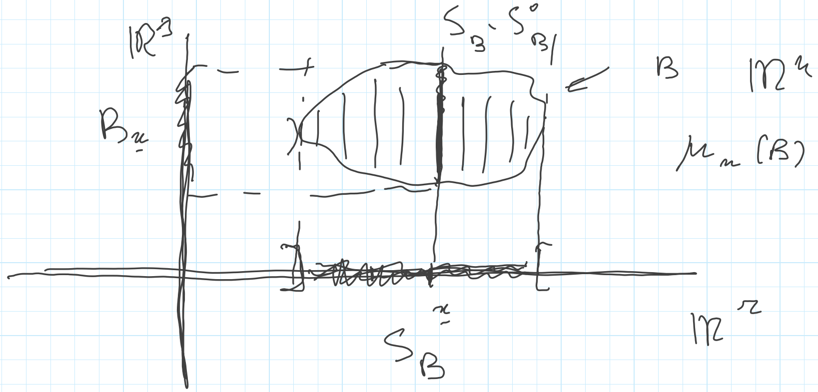
$$S_B \setminus S_B^0 \longrightarrow \overline{\mathbb{R}}$$

$$x \in S_B \setminus S_B^0 \longmapsto \mu_1(B_x)$$

IS MEASURABLE NN FUNCT

AND

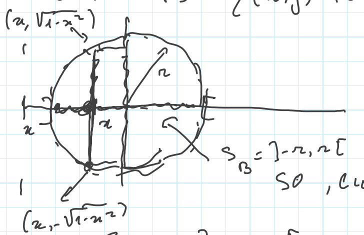
iii) $\mu_n(B) = \int \mu_1(B_x)$!!!



$$\mu_n(B) = \int_{S_B \setminus S_B^0} \mu_1(B_x)$$

$\cap \emptyset$

FIRST EXAMPLE



LET $B = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2 < r^2\}$ $r \in \mathbb{R}^+$ FIXED

LET $\mathbb{R}^2 = \underset{x}{\mathbb{R}^1} \times \underset{y}{\mathbb{R}^1}$,

$S_B =]-r, r[$

SO, CLEARLY

$S =]-r, r[$

AND

$$x \in]-r,r[\ , \quad B_x \overset{\text{DEF}}{=} \{ y \in \mathbb{R} ; \ (x,y) \in B \} =$$

$$= \{ y \in \mathbb{R} : -\sqrt{1-x^2} < y < \sqrt{1-x^2} \}$$

HENCE, FOR $x \in S_B =]-r,r[$, WE HAVE

$$\mu_1(B_x) = \mu_1\left( ]-\sqrt{1-x^2}, \sqrt{1-x^2}[\right) = \underline{2\sqrt{1-x^2}}$$

THEN TONELLI $\implies$

$$\mu_2(B) = \int_{x \in]-r,r[} \mu_1(B_x)\, dx =$$

$$= \int_{]-r,r[} \left(2\sqrt{1-x^2} \right) dx = \pi r^2$$

ANCIENT GREEKS

BREAK QUESTIONS?

BEGIN AGAIN AT 12.20