Monday, February 13, 2023 10:19 AM

BEGIN AT 12.25

**EX 2** LET $n \in \mathbb{Z}^+$

ANY HYPERPLANE, SAY

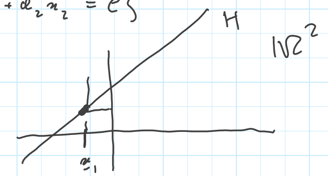
$$H = \{ (x_1, \dots, x_n) \in \mathbb{R}^n ; a_1 x_1 + a_2 x_2 + \dots + a_n x_n = c \} \subseteq \mathbb{R}^n$$

IS S.T. $\mu_n(H) = 0$.

BY INDUCTION ON $n \in \mathbb{Z}^+$.

LET $n = 2$, SO

$$H = \{ (x_1, x_2) \in \mathbb{R}^2 ; a_1 x_1 + a_2 x_2 = c \}$$




SO, SUPPOSE

$a_1 \neq 0$

$$\mathbb{R}^2 = \underset{x_1}{\mathbb{R}^1} \times \underset{x_2}{\mathbb{R}^1}$$

GIVEN $x_1 \in \mathbb{R}^1$

$$B_{x_1} = \{ x_2 \in \mathbb{R}^1 ; a_1 x_1 + b x_2 = c \} = \left\{ x_2 \in \mathbb{R}^1 ; x_2 = \frac{c - a_1 x_1}{b} \right\} = \left\{ \frac{c - a_1 x_1}{b} \right\}$$

$$\forall x_1 \in \mathbb{R}^1, \quad \mu_1(B_{x_1}) = \mu_1\left( \left\{ \frac{c - a_1 x_1}{b} \right\} \right) = 0 \quad (*)$$

THEN

$$S_H \overset{\text{DEF}}{=} \{ x_1 \in \mathbb{R}^1 ; \mu_1^*(B_{x_1}) > 0 \} \overset{(*)}{=} \emptyset$$

$$\mu_2(H) = \int_{S_B = \emptyset} \mu_1(B_{x_1}) = 0$$ QED DONE

NOW, LET $n \in \mathbb{Z}^+$, $n > 2$

$$H = \{ (x_1, \dots, x_n) \in \mathbb{R}^n ; a_1 x_1 + \dots + a_n x_n = c \}$$

SPLIT $\mathbb{R}^n = \mathbb{R}^1 \times \mathbb{R}^{n-1}$

$x_1 \qquad (x_2, x_3, \ldots, x_n)$

FIX $x_1 \in \mathbb{R}^1 \Rightarrow$

$$H_{x_1} = \{ (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} ; \; a_1 x_1 + a_2 x_2 \ldots + a_n x_n = c \}$$

$$= \{ (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} ; \; a_2 x_2 + \ldots + a_n x_n = c_1 - a_1 x_1 \} \subseteq \mathbb{R}^{n-1}$$

IS AN HYPERPLANE IN $\mathbb{R}^{n-1}$

BY INDUCTION HYPOTHESIS

$$\mu_{n-1}(H_{x_1}) = 0 \Rightarrow S_H = \emptyset$$

BY TONELLI,

$$\mu_n(H) = \int_{S_H = \emptyset} \mu_{n-1}(H_{x_1}) = 0 .$$

QED

AS A COROLLARY, WE HAVE

LET $W \subseteq \mathbb{R}^n$ $\quad W$ IS AN "AFFINE SUBSPACE"

(*) $\quad W = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n ; \; \begin{cases} a_{11} x_1 + \ldots a_{1n} x_n = c_1 \\ \ldots \\ a_{n1} x_1 + \ldots + a_{nn} x_n = c_2 \end{cases} \right\}$

CLEARLY, $W \subseteq$ OF SEVERAL HYPERPLANES.

$$W \subseteq H_1 = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n ; \; a_{11} x_1 + \ldots + a_{1n} x_n = c_1 \}$$

BUT $\mu_n(H_1) = 0 \xRightarrow{\text{MONOTONICITY}} \mu_n^*(W) \leq \mu(H_1) = 0$

$\Rightarrow \mu_n^*(W) = 0 \Rightarrow$ $W$ MEAS + $\mu_n(W) = 0$ !!!

RMK $\quad B \subseteq \mathbb{R}^n \quad B$ MEASURABLE $\qquad \mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^3$

BUT, IT MAY HAPPEN, $x \in \mathbb{R}^2$

SUCH THAT $B_x$ (SECTION) NOT MEASURABLE (?)

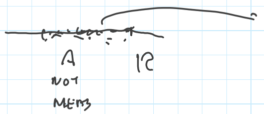
FROM (*) $\quad A \subseteq W \quad W$ SUBSPACE OF $\mathbb{R}^n$

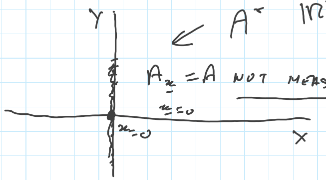
$\dim(W) \leq n$

$$\Rightarrow \left( \mu_n(W) = 0 \Rightarrow \mu_n^*(A) = 0 \Rightarrow A \text{ MEASURABLE} \right\}$$

THE CONVERSE IS FALSE.

LET $A \subseteq \mathbb{R}$, A NOT MEAS !!!




$A^* \quad \mathbb{R}^2$

$A_{x=0} = A$ NOT MEASURABLE!

BUT, IN $\mathbb{R}^2$

$A^*$ IS A SUBSET OF THE VERTICAL AXIS

$\Rightarrow \mu_2(A^*) = 0 \Rightarrow A^*$ MEASURABLE

BUT $x=0$, WHAT IS

$A^*_{x}$

$x=0$

STOP QUESTIONS?

BYEBYE