Monday, February 13, 2023 10:19 AM

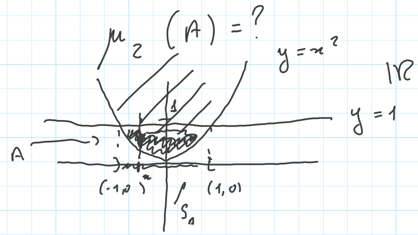
TONELLI THM FOR MEASURES

EX/APPL.

EX 1 $A = \{(x,y) \in \mathbb{R}^2;\ x^2 < y < 1\} \subseteq \mathbb{R}^2$

OPEN $\Rightarrow$ MEAS. $\qquad \Longrightarrow \quad x^2 < 1 \Leftrightarrow |x| = 1$

$\mu_2(A) = ?$



BY TONELLI THM, $\qquad \mathbb{R}^2 = \underset{x}{\underset{\Downarrow}{\mathbb{R}^1}} \times \underset{y}{\underset{\Downarrow}{\mathbb{R}^1}}$

$$\mu_2(A) = \int_{S_A \setminus S_A^0} \mu_1(A_x)\, dx \qquad (*)$$

WHERE $S_A =]-1,1[\ ,\ S_A^0 = \emptyset$.

AND $x \in]-1,1[$

$$A_x = \{y \in \mathbb{R};\ x^2 < y < 1\} =]x^2, 1[$$

HENCE,

$$\mu_1(A_x) = \mu_1(]x^2,1[) = 1 - x^2$$

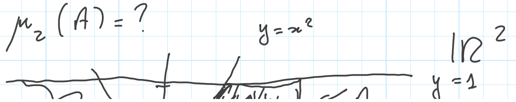
THEN, BY $(*)$

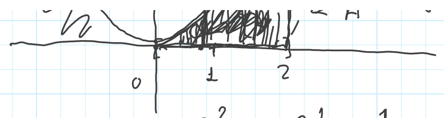
$$\mu_2(A) = \int_{x \in]-1,1[} (1 - x^2)\, dx =$$

$$= R\int_{-1}^{1} (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{x=-1}^{x=1} = \frac{4}{3}. \ \boxed{}$$

EX 2 $A = \{(x,y) \in \mathbb{R}^2;\ 0 < x < 2,\ 0 < y \le x^2,\ 0 < y < 1\}$

$\mu_2(A) = ?$





$x \in S_A = [0,2]$

NOW, SPLIT $\mathbb{R}^2 = \underset{x}{\mathbb{R}^1} \times \underset{y}{\mathbb{R}^1}$

$$\mu_2(A) = \int_{x \in S_A} \mu_1(A_x)\, dx$$

CLEARLY, $S_A^x =]0,2]$ AND,

$\forall x \in S_A^x$ WE HAVE:

$$A_x = \begin{cases} [0, x^2] & \text{IF } 0 \le x \le 1 \\ [0,1] & \text{IF } 1 \le x \le 2 \end{cases}$$

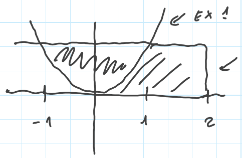
HENCE

$$\mu_1(A_x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 1 & 0 \le x \le 1 \end{cases}$$

$$\mu_2(A) = \int_{]0,2]} \mu_1(A_x)\, dx = \int_{[0,1]} x^2\, dx + \int_{[1,2]} 1\, dx$$

$$= \left[\frac{x^3}{3}\right]_{x=0}^{x=1} + \left[x\right]_{x=1}^{x=2} = \frac{1}{3} + 1 = \frac{4}{3}. \quad \boxtimes$$

(RMK

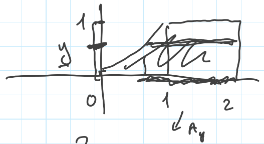


THEY HAVE THE SAME AREA $= 4/3$.)

EX 2 (SECOND WAY)

$$A = \{(x,y) \in \mathbb{R}^2;\ 0 < x < 2,\ 0 < y < x^2,\ y \le 1\}$$

$S_A^y = [0,1]$



A

$\mu_2(A) = ?$ WE SPLIT $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$

$\Leftarrow$ $\quad y \quad x$

$$\mu_2(A) = \int_{y \in S^Y_A = [0,1]} \mu_1(A_y) \, dy \qquad (+)$$

AND $y \in [0,1]$

$$A_y = \{x \in \mathbb{R} ; (x,y) \in A\} = [\sqrt{y}, 2]$$

$$\Rightarrow \mu_1(A_y) = 2 - y^{1/2} \overset{\text{BY } (+)}{\Longrightarrow}$$

$$\Rightarrow \mu_2(A) = \int_{y \in [0,1]} (2 - y^{1/2}) \, dy =$$

$$= \left[2y - \frac{2}{3} y^{3/2}\right]_{y=0}^{y=1} = 2 - \frac{2}{3} = \frac{4}{3}$$

AS EARLIER !! $\square$

EX 3

$$B = \{(x,y) \in \mathbb{R}^3 ; \ x^2+y^2+z^2 < 2^2, \ 3x - 4y + 5z \in \mathbb{R} \setminus \mathbb{Q}\}.$$

i) IS B MEAS? ii) IS B BOREL? iii) IF, YES: COMPUTE $\mu_3(B)$.

YES (i), YES (ii)

SOL. FOR i) & ii).

NOW

$$(\#) \quad B = \{(x,y,z) \in \mathbb{R}^3 ; x^2+y^2+z^2 < 2^2\} \setminus \{(x,y,z) ; 3x - 4y + 5z \in \mathbb{Q}\}$$

THE "BALL" B_2 OF RADIUS 2

OPEN $\Rightarrow$ BORELIAN

BOREL

BOREL AND OF MEASURE 0 $\quad - \bigcup_{q \in \mathbb{Q}}^{\text{COUNT}} \{(x,y,z) \in \mathbb{R}^3 ; 3x - 4y + 5z = q\}$

CLOSED $\Rightarrow$ BOREL & OF MEASURE 0 !! (FROM YESTERDAY)

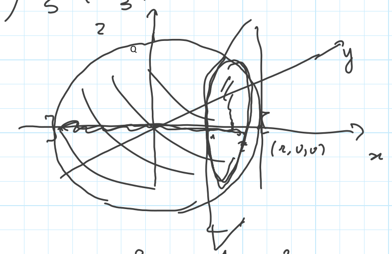
FROM $(\#) \Rightarrow$

$\mu_3(B) = \mu_3(B_2)$ WHERE

$$B_3 = \{(x,y,z) \in \mathbb{R}^3 \; ; \; x^2+y^2+z^2 < r^2\} \qquad r \in \mathbb{R}^+$$

NOW, WE HAVE TO COMPUTE

$$\mu_3(B_3) = ?$$



NOW, SPLIT $\mathbb{R}^3 = \underset{x}{\mathbb{R}^1} \times \underset{(y,z)}{\mathbb{R}^2}$

WE HAVE: IN X $\quad S_A^x =]-r, r[$ AND

FOR $x \in S_A =]-r, r[$

$$A_x = \{(y,z) \in \mathbb{R}^2 ; (x,y,z) \in B_3\} =$$

$$= \{(y,z) \in \mathbb{R}^2 ; y^2+z^2 < r^2 - x^2\} \Rightarrow$$

$$\mu_2(A_x) = \pi(r^2 - x^2) \overset{\text{TONELLI}}{\Longrightarrow}$$

$$\mu_3(B_3) = \int_{x \in S_A =]-r,r[} \pi(r^2 - x^2) \, dx$$

$$= \pi \left[x - \frac{1}{3}x^3 \right]_{x=-r}^{x=r} = \frac{4}{3}\pi r^3 \qquad \boxed{}$$

GREEKS RECOVERED

BREAK QUESTIONS

BEGIN AGAIN A 15.10