EX $A = \{(x,y,z) \in \mathbb{R}^3;\ 0 \leq z \leq x^2 + y^2 + 2,\ |x|<1,\ |y|<1\}$

COMPUTE $\mu_3(A) = ?$

SOL WE SPLIT $\mathbb{R}^3 = \underset{x}{\mathbb{R}^1} \times \underset{(y,z)}{\mathbb{R}^2}$

IN X, $S_A^x =]-1,1[$ AND

$x \in S_A =]-1,1[$ WE HAVE

$$A_x = \{(y,z) \in \mathbb{R}^2;\ 0 \leq z - y^2 \leq 2 + x^2,\ |y| \leq 1\}$$

THEN, TONELLI (FIRST APPL)

$$\mu_3(A) \int_{S_A^x =]-1,1[} \mu_2(A_x)\, dx$$

NOW, GIVEN $x \in]-1,1[$ FIXED, WE HAVE TO COMPUTE

$\mu_2(A_x)$. AGAIN, APPLY TONELLI THM.:

IN Y, $y \in S_{A_x} =]-1,1]$, AS A MATTER OF FACTS

$$(A_x)_y = \{z \in \mathbb{R};\ (x,y,z) \in A\}$$

$$= \{z \in \mathbb{R};\ 0 \leq z \leq \underbrace{2 + x^2 + y^2}_{|y|<1}\},\quad |y|<1$$

$$\mu_3(A) = \int_{S_A^x} \mu_2(A_x)\, dx =$$

$$= \int_{x \in S_A^x} \Big(\int_{y \in S_{A_x}^y} \mu_1((A_x)_y)\, dy \Big)\, dx \qquad (*)$$

$\uparrow$ FUNCT IN y

IS A FUNCT. IN z

$$\mu_1\left((A_x)_y \right) = \mu_1\left(\{ z \in \mathbb{R};\ 0 < z < 2 + x^2 + y^2 \} \right) =$$

$$= 2 + x^2 + y^2$$

$$\mu_3(A) = \int_{x \in]-1,1[} \left(\int_{y \in]-1,1[} (2 + x^2 + y^2)\, dy \right) dx$$

$$= \int_{x \in]-1,1[} \left(\left[(2 + x^2) y + \frac{y^3}{3} \right]_{y=-1}^{y=1} \right) dx$$

$$= \int_{x \in]-1,1[} \left(2(2 + x^2) + \frac{2}{3} \right) dx$$

AS A FUNCT IN x

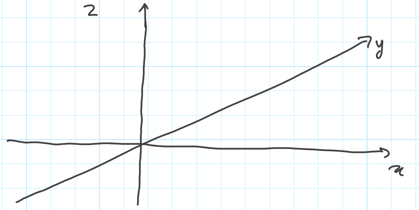
$$= 2\left[\frac{x^3}{3} + 2x \right]_{x=-1}^{x=1} + \frac{2}{3} \left[x \right]_{x=-1}^{x=1} = \frac{32}{3}$$

FROM GEOMETRIC POINT OF VIEW:

RECALL

$$A = \{ (x,y,z) \in \mathbb{R}^3;\ 0 \le z \le x^2 + y^2 + 2,\ |x| \le 1, |y| \le 1 \}.$$

WHAT IS A?



$\mathbb{R}^3$

WHAT IS $z = x^2 + y^2 + 2$ $(+)$

BUT $x^2 + y^2 = \|(x,y)\|^2$

$\|(x,y)\| = x^2 + y^2$

$\Downarrow$

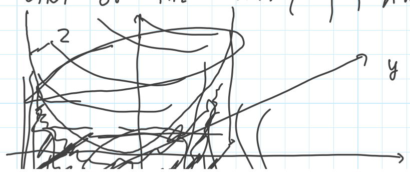
$d((x,y),(0,0))$ FIXED

POINT OF THE FORM $(+)$ ARE

$z = x^2 + y^2$

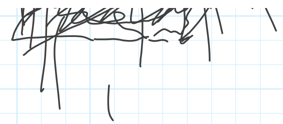
$\Downarrow$

$z \le x^2 + y^2$



$\times r$

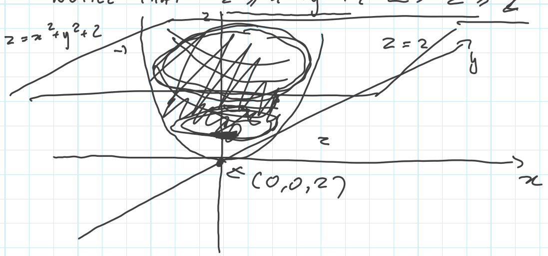
0 .



BUT WE HAVE ALSO $|x|<1, |y|<1$

EX $\quad A = \{ (x,y,z) \in \mathbb{R}^3 ; \ z \geq x^2+y^2+2 \ , \ z < 4 \}$

NOTICE THAT $\underline{z \geq x^2+y^2+2} \implies z \geq 2$



$\mathbb{R}^3$

WE SPLIT $\mathbb{R}^3 = \underset{z}{\mathbb{R}^1} \times \underset{(x,y)}{\mathbb{R}^2}$

IT IS CLEAR THAT, IN Z THE SUPPORT IS

$z \in S_A^z = [2,4]$ AND FOR $z \in S_A^z = [2,4]$

$$\mu_3(A) = \int_{S_A^z} \mu_2(A_z) \, dz$$

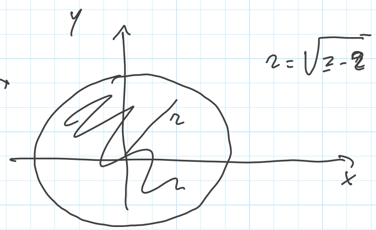
WHERE, GIVEN $z \in [2,4]$

$$A_z = \{ (x,y) \in \mathbb{R}^2 ; \ (x,y,z) \in A \}$$

$$= \{ (x,y) \in \mathbb{R}^2 ; \ x^2+y^2 \leq z-2 \}$$

IN PLAIN WORDS

$A_z \to$



$r = \sqrt{z-2}$

HENCE FOR $z \in S_A^2 =]2,4[$

$$\mu_3(A) = \int_{z \in S_A^2 =]2,4[} \mu_1(A_z) \, dz =$$

$$= \int_{S_A^2 =]2,4[} \pi(z-2)\, dz = \pi \left[\frac{z^2}{2} - 2z \right]_{z=2}^{z=4}$$

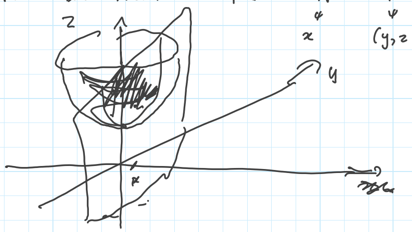
$$= \pi \left[\frac{16}{2} - \frac{4}{2} - 8 + 4 \right] = 2\pi \quad !!!$$

$\overline{6} - 8 + 4$?

OK

BEST WAY

IF WE SPLIT $\mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$
$x \quad (y,z)$



$$\mu_3(A) = \int_{x \in S_A^x} \mu_2(A_x) \, dx$$

$$= \int_{x \in S_A^x} \left(\int_{y \in S_{A_x}^Y} \mu_1((A_x)_y) \, dy \right) dz$$

STOP QUESTIONS?

BYE BYE