Monday, February 13, 2023 10:20 AM

$B = \{(x,y,z) \in \mathbb{R}^3 ;\ 0 < z < -x^2 - y^2 + 1 ,\quad y - x^2 \in \mathbb{R} \setminus \mathbb{Q}\}$

i) B MEAS? ii) B BOREL? iii) IF YES, COMPUTE $\mu_3(B)$.

YES   YES   ???

QUESTION i) & ii):

WE HAVE:

$$B = \underbrace{\{(x,y,z) \in \mathbb{R}^3 ;\ 0 < z < -x^2 - y^2 + 1\}}_{=A\ \text{(OPEN} \Rightarrow \text{BOREL)}} \setminus \underbrace{\{(x,y,z) \in \mathbb{R}^3 ;\ y - x^2 \in \mathbb{Q}\}}_{=C\ \text{BOREL}}$$

B BOREL $\Downarrow$ B MEAS

NOW $C = \{(x,y,z) \in \mathbb{R}^3 ;\ y - x^2 \in \mathbb{Q}\}$ BOREL $\Uparrow$

$$= \bigcup_{q \in \mathbb{Q}}^{\text{COUNT.}} \{(x,y,z) \in \mathbb{R}^3 ;\ y - x^2 = q\} \text{ BOREL}$$

$\qquad = C_q$ CLOSED $\Rightarrow$ BOREL $\quad q \in \mathbb{Q}$

CLEARLY, SINCE $B = A \setminus C \Rightarrow \mu_3(B) = \mu_3(A) - \mu_3(C)$ ↑ ???

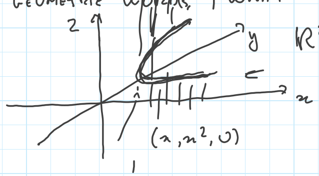
SUBPROBLEM 1) COMPUTE

$$\mu_3(C) = \mu_3\Big(\bigcup_{q \in \mathbb{Q}}^{\text{COUNT}} C_q\Big) \overset{\text{ADDITIVITY}}{=} \sum_{q \in \mathbb{Q}} \mu_3(C_q)$$

SUBPROBLEM 2) GIVEN $q \in \mathbb{Q}$, COMPUTE

$$\mu_3(C_q) = \{(x,y,z) \in \mathbb{R}^3 ;\ y - x^2 = q\}$$

IN GEOMETRIC WORDS, WHAT C_q ($q \in \mathbb{Q}$ FIXED) ???



$\mathbb{R}^3$

INTERSECTION WITH THE XY

$y = x^2 + q$

$\mu_3(C_q) \overset{\text{TONELLI}}{=}$

WE SPLIT $\quad q \in \mathbb{Q}$ FIXED

$\mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$

(*)

$$= \int_{x \in S^x_{C_q}} \mu_2 \underset{0}{\underset{\shortparallel}{((C_q)_x)}} \, dx \quad .. \quad x \quad (y,z)$$

NOW $S^x_{C_q} = \mathbb{R}$ AND GIVEN

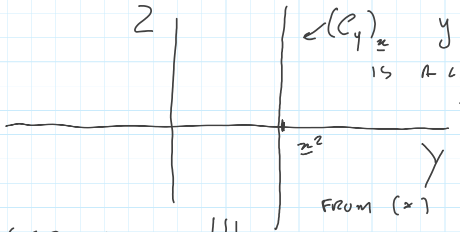
$\underline{x} \in S^x_{C_q} = \mathbb{R}$ WE HAVE :

$$(C_q)_{\underline{x}} = \{ (y,z) \in \mathbb{R}^2 ; (\underline{x}, y, z) \in C_q \}$$

$$= \{ (y,z) \in \mathbb{R}^2 ; y = \underline{x}^2 \} =$$

$$= \{ (y,z) \in \mathbb{R}^2 ; y = \underset{\uparrow \text{CUST}}{\underline{x}}^2 \}$$

$(C_q)_{\underline{x}}$



$\leftarrow (C_q)_{\underline{x}}$ $y = \underline{x}^2$. IS A LINE IN $\mathbb{R}^2$: YZ

FROM $(*)$

$$\Rightarrow \mu_2((C_q)_{\underline{x}}) = 0 \; !!! \quad \Longrightarrow$$

$$\mu_3(C_q) \overset{\text{TONELLI}}{=} \int_{x \in S^x_{C_q}} \mu_2 \underset{0}{\underset{\shortparallel}{((C_q)_{\underline{x}})}} \, d\underline{x} = 0 \; !!!$$

THEN , FROM $B = A - C \Rightarrow$

$$\mu_3(B) = \mu_3(A) - \mu_3 \underset{0}{\underset{\shortparallel}{(C)}} \qquad (\dagger)$$

$$\mu_3(C) = \sum_{q \in \mathbb{Q}} \mu_3 \underset{0}{\underset{\shortparallel}{(C_q)}} = 0 \qquad q \in \mathbb{Q}$$

$$\mu_3(B) = \mu_3(A) \quad !!!$$

WHERE , REMEMBER

[illegible]

$A = \{(x,y,z) \in \mathbb{R}^3 \; ; \; 0 < z < -x^2-y^2+1\}$ $\Leftarrow (+)$

COMPUTE $\mu_3(A)$!!!

WE SPLIT

$\mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$
$\quad\quad\quad\; x \quad\;\; (y,z)$

$$\mu_3(A) \overset{\text{TONELLI}}{=} \int_{x \in S_A^x} \mu_2(A_x) \, dx$$

WHAT IS $S_A^x =]-1,1[$

FOR $x \in]-1,1[$, COMPUTE

WE SPLIT

$\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$
$\quad\quad\quad\; y \quad\;\; z$

$$(\#) \quad \mu_2(A_x) \overset{\text{TONELLI}}{=} \int_{y \in S^y_{A_x}} \mu_1((A_x)_y) \, dy$$

WHERE (RECALL $A_x = \{(y,z) \; ; \; 0 < z + y^2 < \underbrace{1-x^2}_{\text{CONST}}\}$)

$$(A_x)_y = \{z \in \mathbb{R} : 0 < z < 1 - x^2 - y^2\} =$$

$$=]\,0, 1-x^2-y^2\,[\; \Rightarrow \mu_1((A_x)_y) = 1 - x^2 - y^2$$

AND $S_{A_x} = \{y \in \mathbb{R} ; |y| < \sqrt{1-x^2}\}$

SO, IN CONCLUSION:

$$\mu_2(A_x) = \int_{y \in S_{A_x}} \mu_1((A_x)_y) \, dy =$$

$$= \int_{y \in]-\sqrt{1-x^2}, \sqrt{1-x^2}[} (1 - x^2 - y^2) \, dy$$

AS A FUNCTION OF y

$\Rightarrow$

$$\mu_3(A) = \int_{x \in S_A^x} (\mu_2(A_x)) \, dx$$

$$= \int \left(\int (1-x^2-y^2) \, dy \right) dx$$

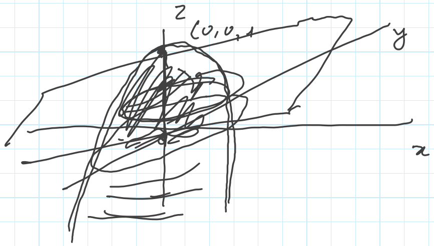
$x \in S^x_A \qquad y \in S^y_{A_x}$ ← FUNCT IN y

IS FUNCT OF x

IS A FUNCT OF x

RMK $\qquad A = \{ (x,y,z) \in \mathbb{R}^3 \,;\, 0 < z < -x^2 - y^2 + 1 \}$

IN A GEOMETRIC WAY, WHAT IS A?



$\leftarrow z = -x^2 - y^2 + 1$???

$-x^2 - y^2 =$

$= -\|(x,y)\|^2$

z FIXED

A_z

GIVEN $\quad z \in S^z_A =]0,1[$

$A_z = \{ (x,y) \in \mathbb{R}^2 \,;\, x^2 + y^2 = 1 - z \}$ CIRCLE

$\mu_2(A_z) = \pi(1-z) \Longrightarrow$

$\mu_3(A) \overset{\text{TONELLI}}{=}$

$$= \int_{z \in S^z_A =]0,1[} \pi(1-z) \, dz$$

MUCH SIMPLER !!!

WE SPLIT

$\mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$

$\qquad\quad \uparrow \qquad\; \shortparallel$

$\qquad\quad z \qquad (x,y)$

BREAK $\qquad$ QUESTIONS?

BEGIN AGAIN AT 12.15