Monday, February 13, 2023 10:21 AM

TONELLI THM FOR INTEGRALS

THM LET $f : A \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE + NON NEGATIVE !!!

$$\boxed{\begin{array}{c} \mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^3 \\ 2+3=n \end{array}}$$

AS USUAL:

LET 1) $\underline{x} \in \mathbb{R}^2$

$$A_{\underline{x}} = \{ y \in \mathbb{R}^3 \, ; \, (\underline{x}, y) \in A \}$$

SECTION

2) $S_A = \{ x \in \mathbb{R}^2 \, ; \, \mu_3^*(A_x) > 0 \}$ SUPPORT

3) $S_A^0 = \{ x \in \mathbb{R}^2 \, ; \, A_x \text{ NOT MEAS} \}$ BAD SUPPORT

RECALL THAT:

i) S_A, S_A^0 MEASURABLE IN $\mathbb{R}^2$

ii) $\mu_2(S_A^0) = 0$!!!

NOW:

$(\underline{x}, \underline{y}) \in \mathbb{R}^n$

$\underline{x} \in \mathbb{R}^2, \; y \in \mathbb{R}^3$

1) $\forall \underline{x} \in S_A \setminus S_A^0$ THE FUNCT

$$f_{\underline{x}}(y) = f(\underline{x}, y) : A_{\underline{x}} \to \overline{\mathbb{R}}$$

IS MEAS NN !!!

(NOTICE THAT $A_{\underline{x}}$ MEAS $\forall \underline{x} \in S_A \setminus S_A^0$.)

2) $f_1 : S_A \setminus S_A^0 \to \overline{\mathbb{R}}$

$$f_1 : \underline{x} \in S_A \setminus S_A^0 \longrightarrow \int_{A_{\underline{x}}} f_{\underline{x}}(y) \, dy .$$

IS MEAS NN !!!

THEN

$$\int_A f = \int_{S_A \setminus S_A^0} f_1(\underline{x}) \, d\underline{x} =$$

$$= \int \left(\int f_{\underline{x}}(y) \, dy \right) d\underline{x}$$

$$\int_{S_A \setminus S_A^\circ} \ldots_{y \in A_x}$$

$$= \int_{S_A \setminus S_A^\circ} \underbrace{\Big(\int_{y \in A_x} f(x,y)\, dy\Big)}_{\text{IS A FUNCT IN } x} dx$$

$\Leftarrow$ FUNCT IN y

FUBINI THM : IS THE CASE

$$f : B \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$$

WHERE HP) NN IS DROPPED !

BUT HP) f IS SUMMABLE IS ADDED !!!

$$\int_B f < +\infty$$

AND THE ASSERTION IS THE SAME AS THAT OF TONELLI THM !

EX $A = \{(x,y) \in \mathbb{R}^2 ;\ |x| < 1,\ |y| < 1\}$

LET $f : A \to \mathbb{R}$, $f(x,y) = x + y$

WE WANT TO COMPUTE $\int_A f = \int f(x,y)\, dx\, dy$!!!

BUT f IS NOT NN ON A. SO WE HAVE TO APPLY FUBINI VERSION !!! HENCE, WE HAVE THAT $-\infty < \int_A f < +\infty$!!! WHY?

LET $\overline{A} = \{(x,y) \in \mathbb{R}^2 ;\ |x| \le 1,\ |y| \le 1\}$ $\Leftarrow$ CLOSED + LIMITED $\Downarrow$ COMPACT !

CLOSURE OF A

AND CONSIDER $f : \overline{A} \to \mathbb{R}$; $f(x,y) = x + y$ $\Leftarrow$ CONT $\forall (x,y) \in \overline{A}$.

SO $f(x,y) = x + y$ IS CONT OVER THE COMPACT SET $\overline{A}$.

THE WEIERSTASS THM HOLDS, THAT:

TH $\Longrightarrow$

$$\exists\ M = \max \{ f(x,y) ; (x,y) \in \overline{A} \} \in \mathbb{R}$$

$$\exists\ m = \min \{ f(x,y) ; (x,y) \in \overline{A} \} \in \mathbb{R}$$

$$\Downarrow$$

$$m \cdot \mu(\bar{A}) \le \int_{\bar{A}} f \le M \cdot \mu(\bar{A})$$

BUT $\bar{A}$ LIMITED

$$\Downarrow$$

$\mu(\bar{A}) < +\infty$

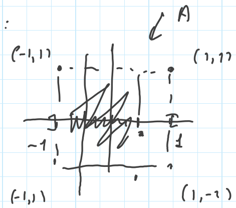
$$\Downarrow$$

$$-\infty < \int_A f < +\infty$$

SO, WE CAN USE FUBINI THM:

$$S_A =]-1,1[$$

$$\mathbb{R}^2 = \underset{x}{\mathbb{R}^1} \times \underset{y}{\mathbb{R}^1}$$



FUBINI $\Rightarrow$

$$\int_A f = \int_{x \in S_A^x =]-1,1[} \left(\int_{A_x} f(x,y)\,dy \right) dx$$

$$= \int_{S_A^x =]-1,1[} \left(\int_{A_x =]-1,1[} (x+y)\,dy \right) dx$$

$$= \int_{S_A^x =]-1,1[} \left(\left[xy + \frac{y^2}{2} \right]_{y=-1}^{y=1} \right) dx$$

$$= \int_{x \in]-1,1[} 2x\,dx = \left[x^2\right]_{x=-1}^{x=1} = 0. \ !!!$$

EX/NOTATION $\quad A \subseteq \mathbb{R}^3$

$$f: A \to \bar{\mathbb{R}}, \quad f(x,y,z)$$

HOW TO EXPRESS

$$\int_A f = \int f(x,y,z)\,dz\,dy\,dx \ ???$$

TONELLI/FUBINI

$$\Rightarrow \int f(x,y,z)\,dz\,dy\,dx =$$

$$= \int_{x \in S^x_A} \underbrace{\left(\int_{(y,z) \in A_x} f(x,y,z)\, dz\, dy \right)} dx$$

NOW (TONELLI/FUBINI)

$$\int_{(y,z) \in A_x} f(x,y,z)\, dz\, dy = \int_{y \in S^y_{A_x}} \left(\int_{z \in (A_x)_y} f(x,y,z)\, dz \right) dy$$

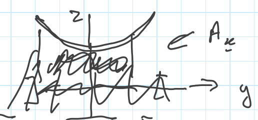
$$= \int_{x \in S^x_A} \underbrace{\left(\int_{y \in S^y_{A_x}} \underbrace{\left(\int_{(A_x)_y} \underbrace{f(x,y,z)}_{\text{FUNCT IN } z} dz \right)}_{\text{FUNCT IN } y} dy \right)}_{\text{IS A FUNCTION IN } x} dx$$

EX

$$A = \{ (x,y,z) \in \mathbb{R}^3 : 0 < z < x^2+y^2+2 \ ,\ x^2+y^2 < 1 \}$$

$$f(x,y,z) = x+y+z$$

$$S^x_A =]-1,1[\quad \& \quad A_x = \{ (y,z) \in \mathbb{R}^2 ;\ 0 < z - y^2 \le x^2+2 \ ,\ y^2 < 1-x^2 \}$$



YZ

$$y \in S^y_{A_x} = \left] -\sqrt{1-x^2}, \sqrt{1-x^2} \right[$$

AND $(A_x)_y = \{ z \in \mathbb{R} : 0 < z < x^2+y^2+2 \}$

$$\int_A f(x,y,z)\, dz\, dy\, dx =$$

$$= \int_{-1}^{1} \underbrace{\left(\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \underbrace{\left(\int_0^{x^2+y^2+2} \underbrace{(x+y+z)}_{\text{FUNCT IN } z} dz \right)}_{\text{FUNCT IN } y} \right) dy \right)}_{\text{FUNCT IN } x} dx$$

THE <u>COURSE IS OVER</u>