Tuesday, May 16, 2023 10:21 AM

INIZIO ORE 15.15

TEOREMA DI INVERSIONE (ROTA, 1964)

SIA $(\underline{P}, \leq)$ POSET LOC. FINITO

SIANO $f, g : \underline{P} \to \mathbb{R}$ t.e.

$$\sum_{x \leq y} f(x) = g(y) \quad \forall y \in P \quad \text{(REL. DIRETTA)} \; (*)$$

$$\Big\Updownarrow$$

$\forall y \in \underline{P}$ SI HA:

$$f(y) = \sum_{x \leq y} \mu_{\underline{P}}(x, y)\, g(x) \quad !!!$$

PESATA — SOMME PARZIALI (REL. INVERSA) $(**)$

FATTO 1 SIA S INSIEME, CONSIDERIAMO

$$\left(\mathbb{P}(S), \subseteq \right) \quad (\text{RETICOLO})$$

INCLUSIONE

LATTICE

CHI È LA SUA FUNZIONE DI MÖBIUS? $\mu : \mathbb{P}(S) \times \mathbb{P}(S) \to \mathbb{Z}$

THM DATI $A, B \in \mathbb{P}(S)$ $(A, B \subseteq S)$

CON $A \subseteq B$ SI HA:

$$\mu(A, B) \stackrel{!}{=} (-1)^{|B|-|A|}$$

?!?!

PROOF

DOBBIAMO DIMOSTRARE CHE (3):

DATI $A, B \subseteq S$, $A \subseteq B$, $A \neq B$

SI HA

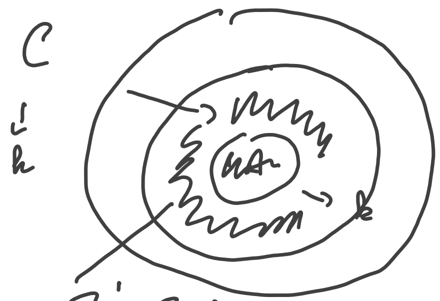
$$(+) \quad \sum_{A \subseteq C \subseteq B} (-1)^{|B|-|C|} = 0 \quad (\text{DA FARE})$$

MA SIA $|A| = k$, $|B| = m$ $\quad (A \subsetneq B \Rightarrow k < m)$

$$(+) \Longleftrightarrow \sum_{h=k}^{m} \sum_{\substack{A \subseteq C \subseteq B \\ |C| = h}} (-1)^{m-h} =$$

$$= \sum_{h=k}^{m} \binom{m-k}{h-k} (-1)^{m-h} =$$

$$= \sum_{j=0}^{m-k} \binom{m-k}{j} (-1)^{m-k-j} = 0 \;!!!$$



$C' = C \setminus A$

$|C'| = h - k$

$$(-1)^{|B|-|A|} \stackrel{!}{=} \mu(A,B) \qquad \text{PER } A \subseteq B \subseteq S$$

FUNZ. DI MÖBIUS DI

$(\mathbb{P}(S), \subseteq)$!!!

DETERMINARE :

$$\# \{ F : \underline{k} \xrightarrow{SU} \underline{n} \} = ?$$

DATO $A \subseteq \underline{n}$ SIA

$$(*) \begin{cases} f(A) = \# \{ F : \underline{k} \to \underline{n} \, ; \, \operatorname{Im} F = A \} \\ f : \mathbb{P}(S) \to \mathbb{R} \end{cases}$$

CIOÈ F È "SURIETTIVA" "SU $A \subseteq \underline{n}$"

DATO $B \subseteq \underline{n}$, SIA

$$g(B) = \# \{ F : \underline{k} \to \underline{n} \, ; \, \operatorname{Im} F \subseteq B \}$$

$$g : \mathbb{P}(S) \to \mathbb{R}.$$

ORA: $B \subseteq \underline{n}$

$$\{ F : \underline{k} \to \underline{n} \, ; \, \operatorname{Im} F \subseteq B \} =$$

$$= \dot{\bigcup_{A \subseteq B}} \{ F : \underline{k} \to \underline{n} \, ; \, \operatorname{Im} F = A \}$$

$$\Downarrow$$

$$\underline{g(B)} = \# \{ F : \underline{k} \to \underline{n} ; \ Im F \overset{\vee}{\subseteq} B \} =$$

$$= \sum_{A \subseteq B} \# \{ F : \underline{k} \to \underline{n} ; \ Im F = A \}$$

$$= \underline{\sum_{A \subseteq B} f(A)}$$

CIOÈ
$\forall B \subseteq \underline{n}$

$$g(B) = \sum_{A \subseteq B} f(A) \qquad \forall B \in \mathbb{P}(S) \ (B \subseteq S)$$

REL. DIRETTA

$\Downarrow$ INVERSIONE DI MOEBIUS

$$f(B) = \sum_{A \subseteq B} \mu(A, B) \, g(A) .$$

RICORDANDO

$|B|-|A|$

$\mu(A,B) = (-1)$

$A \subseteq B$

$$f(B) = \sum_{\substack{A \\ A \subseteq B}} (-1)^{|B|-|A|} \, g(A) \qquad (\times \times)$$

PER $B = \underline{n}$

$$\#\{F : \underline{k} \xrightarrow{SU} \underline{n}\} = = f(\underline{n}) = f(B) \overset{(\times\times)}{=}$$

$$= \sum_{A \subseteq \underline{n}} (-1)^{n-|A|} g(A) =$$

$$= \sum_{j=0}^{n} \Big(\sum_{\substack{A \subseteq \underline{n} \\ |A| = j}} (-1)^{n-j} g(A) \Big) =$$

$$g(A) = \#\{F : \underline{k} \to \underline{n} ; \ \mathrm{Im}\, F \subseteq A\} \qquad |A| = j$$

$$= j^k$$

$$= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^k$$

THM

$$\# \{ F : \underline{k} \xrightarrow{SU} \underline{n} \} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^k \quad ?!!$$

BREAK QUESTIONS?