Tuesday, May 16, 2023 10:22 AM

RICORDO

THM (PRINCIPIO DI INVERSIONE / CASO INSIEMISTICO) . (PRIMA VERSIONE)

S INS FINITO, POSET $(\mathbb{P}(S), \subseteq)$.

INS. DELLE PARTI — ORDINE DI INCLUSIONE

SIANO $f, g : \mathbb{P}(S) \to \mathbb{R}$

t.c.

$$\forall B \subseteq S \quad \sum_{A \subseteq B} f(A) = g(B) \qquad \text{(REL. DIRETTA)}$$

$$\Updownarrow$$

$$\forall B \subseteq S \quad f(B) = \sum_{A \subseteq B} (-1)^{|B|-|A|} g(A) \qquad \text{(REL. INVERSA)}.$$

APPL.

$$\#\{f : \underline{k} \xrightarrow{su} \underline{n}\} =$$

$$= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^k .$$

THM (PRINCIPIO DI INVERSIONE / INSIEMISTICO) (FORMA DUALE)

CIOÈ, IL POSET SARÀ $(\mathbb{P}(S), \supseteq)$?

INS PART — ORDINE "INVERSO"

SIA $f, g : \mathbb{P}(S) \to \mathbb{R}$ t.c.

$$\forall B \subseteq S \quad \sum_{A \supseteq B} f(A) = g(B) \qquad \text{(REL. DIRETTA)}$$

$$\Downarrow$$

$$\forall B \subseteq S \quad f(B) = \sum_{A \supseteq B} (-1)^{|A|-|B|} g(B) \qquad \text{(REL. INVERSA)}$$

APPL PROBLEMA DEI DERANGEMENT GENERALIZZATI

$n, k \in \mathbb{N}$

$d_{n,k} = \#$ n-PERMUTAZIONI $\left(\sigma : \underline{n} \xrightarrow[1-1]{su} \underline{n}\right)$

che abbiano esattamente
k punti fissi

(Ricordo: $i \in \underline{n}$ è punto fisso $\iff \sigma(i) = i$).

(Ricordo 2) $\underline{\underline{d_{n,k}}} = \binom{n}{k} d_{n-k}$ ← $(n-k)$-permutazione senza punti fissi)

Forma chiusa per $d_{n,k}$ $, \forall n,k$.

Procediamo per inversione di Möbius "duale"

cioè $(\mathbb{P}(\underline{n}), \supseteq)$ $\mathbb{P}\mathbb{P}$ $S = \underline{n} = \{1,2,\dots,n\}$

ins parti, inclusione

$f, g : \mathbb{P}(\underline{n}) \to \mathbb{R}$

$$\forall A \subseteq \underline{n} \quad f(A) \overset{\text{def}}{=} \# \{ \sigma : \underline{n} \xrightarrow[1-1]{su} \underline{n} \ ; \ FIX(\sigma) = A \}$$

$$\forall B \subseteq \underline{n} \quad g(B) \overset{\text{def}}{=} \# \{ \sigma : \underline{n} \xrightarrow[1-1]{su} \underline{n} \ ; \ FIX(\sigma) \supseteq B \}$$

Chiaramente: $\forall B \subseteq \underline{n}$

$$\{ \sigma : \underline{n} \xrightarrow[1-1]{su} \underline{n} \ ; \ FIX(\sigma) \supseteq B \} \overset{!}{=} \dot{\bigcup_{A \supseteq B}} \{ \sigma : \underline{n} \xrightarrow[1-1]{su} \underline{n} \ ; \ FIX(\sigma) = A \} \quad (*)$$

$$\# \{ \sigma : \underline{n} \xrightarrow[1-1]{su} \underline{n} \ ; \ FIX(\sigma) \supseteq B \} = g(B)$$

$(*)$ $$\sum_{A \supseteq B} \# \{ \sigma : \underline{n} \xrightarrow[1-1]{su} \underline{n} \ ; \ FIX(\sigma) = A \} = \sum_{A \supseteq B} f(A)$$ Rel. diretta

Inversione duale

$$\forall B \subseteq \underline{n} \quad f(B) \overset{\text{Thm}}{=} \sum_{A \supseteq B} (-1)^{|A|-|B|} g(A) =$$

Sia

$|B| = k$

$$= \sum_{h=k}^{n} \Big(\sum_{A,\ |A|=h,\ A \supseteq B} (-1)^{h-k} g(A) \Big)$$

VALORE È SEMPRE $(n-h)!$

$$= \sum_{h=k}^{n} (-1)^{h-k} \binom{n-k}{h-k} (n-h)! =$$

$$= \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-k)!}{(h-k)!\,\cancel{(n-h)!}} \cancel{(n-h)!}$$

$$= \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-k)!}{(h-k)!} \quad (*).$$

$\# \{ \sigma : \underline{n} \xrightarrow[\text{bij}]{\text{su}} \underline{n} \ ;\ \mathrm{FIX}(\sigma) = B \}$ PER UN $B \subseteq \underline{n}$ FISSATO t.c. $|B| = k$.

$\Downarrow$ $(*)$

$$\underline{d_{n,k}} = \binom{n}{k} \cdot \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-k)!}{(h-k)!}$$

$\#$ SCELTE DI $B \subseteq \underline{n}$, $|B| = k$

$$= \frac{n!}{k!\,\cancel{(n-k)!}} \sum_{h=k}^{n} (-1)^{h-k} \frac{\cancel{(n-k)!}}{(h-k)!}$$

$$= \frac{n!}{k!} \sum_{h=k}^{n} \frac{(-1)^{h-k}}{(h-k)!} \qquad \underline{\text{TUM}}$$

CASO PART (DERANGEMENT $\iff k = 0$)

FORMA CHIUSA $\rightarrow$ $d_n = n! \sum_{h=0}^{n} \frac{(-1)^h}{h!}$!!!

RICORDO
$d_1 = 0$, $d_2 = 1$
$d_n = (n-1)(d_{n-2} + d_{n-1})$

PROBABILISTICA $\quad \mathbb{P}_n$ = PROBABILITÀ CHE UNA n-PERMUTAZIONE NON ABBIA PUNTI FISSI

ORA, OVVIAMENTE:

$$\mathbb{P}_n = \frac{d_n}{n!} = \sum_{h=0}^{n} \frac{(-1)^h}{h!} \qquad (**)$$

CONSIDERO "L'ASINTOTICA"

$$P_n \xrightarrow[n\to\infty]{} P_\infty \qquad ???$$

QUINDI

$$(+) \quad \begin{cases} P_\infty \overset{def}{=} \lim_{n\to\infty} P_n \overset{!}{=} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \leftarrow \text{QUANTO VALE ???} \\ \qquad \Downarrow \\ = \dfrac{1}{e} = e^{-1} \end{cases}$$

$\uparrow$ NUMERO DI NEPERO / EULERO

PERCHÈ? INFATTI RICORDO (FORMULA DI TAYLOR).

$$e^x \underset{\text{TAYLOR}}{=} \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad (++)$$

VALUTANDO (++) PER $x = -1$

$$\Downarrow$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e}. \quad \blacksquare$$

BREAK QUESTIONS?

INIZIO ORE 10.25