Tuesday, May 16, 2023 10:23 AM

PRINCIPIO INCLUSIONE/ESCLUSIONE

DATI SONO

i) Ω INSIEME (FINITO) "SPAZIO CAMPIONE"

ii) $A_1, A_2, \dots, A_n \subseteq \Omega$ "EVENTI PROIBITI"

iii) $\underline{n} = \{1, 2, \dots n\}$.

RICORDO:

1) PRODOTTO COMPLETO SIA $T \subseteq \underline{n}$ IL PROD. COMPL. RELATIVA A $T \subseteq \underline{n}$ È:

$$\bigcap_{i \in T} A_i \cap \bigcap_{i \notin T} A_i^c = \qquad \subseteq \Omega$$

$$= \{x \in \Omega;\ x \in A_i\ \forall i \in T\ \&\ x \notin A_i,\ i \notin T\}.$$

2) POI, DATO $S \subseteq \underline{n}$, SI CONSIDERI "INTERSEZIONI SEMPLICI"

$$\left| \bigcap_{i \in S} A_i \right| .$$

FATTO FONDAMENTALE.

DATO $S \subseteq \underline{n}$

$$(*) \quad \bigcap_{i \in S} A_i \overset{!}{=} \dot{\bigcup_{\substack{T \subseteq \underline{n} \\ T \supseteq S}}} \left(\bigcap_{i \in T} A_i \cap \bigcap_{i \notin T} A_i^c \right)$$

ALLORA, POSSIAMO DEFINIRE DUE FUNZIONI

$$f, g : \mathbb{P}(\underline{n}) \longrightarrow \mathbb{R} \quad \text{OVE}$$

OVE:

i) PER $T \subseteq \underline{n}$

$$f(T) = \#\left(\bigcap_{i \in T} A_i \cap \bigcap_{i \notin T} A_i^c\right) = \left| \bigcap_{i \in T} A_i \cap \bigcap_{i \notin T} A_i^c \right|$$

ii) PER $S \subseteq \underline{n}$; $g(S) = \# \bigcap_{i \in S} A_i = \left| \bigcap_{i \in S} A_i \right|$

PASSANDO ALLE CARDINALITÀ

$\forall S \subseteq \underline{n}$ $\quad g(S) = \left|\bigcap_{i\in S} A_i\right| \doteq$
$S \in \mathcal{P}(\underline{n})$

$$= \sum_{\substack{T\subseteq \underline{n}\\ T\supseteq S}} f(T) =$$ REL. DIRETTA

$$\overset{\text{DEF}}{=} \sum_{\substack{T\subseteq \underline{n}\\ T\supseteq S}} \left|\bigcap_{i\in T} A_i \cap \bigcap_{i\notin T} A_i^c\right|$$

$\Downarrow$ INV. MÖBIUS "FORMA DUALE"

$$\forall S \subseteq \underline{n} \quad f(S) \overset{\text{DEF}}{=} \left|\bigcap_{i\in S} A_i \cap \bigcap_{i\notin S} A_i^c\right| =$$

$$= \sum_{\substack{T\subseteq \underline{n}\\ T\supseteq S}} (-1)^{|T|-|S|} g(T) \overset{\text{DEF}}{=}$$

$$= \sum_{\substack{T\subseteq \underline{n}\\ T\supseteq S}} (-1)^{|T|-|S|} \left|\bigcap_{i\in T} A_i\right|$$

$(\ast\ast)$

SE $S=\emptyset$, COSA DIVENTA $(\ast\ast)$???

$$f(\emptyset) = \sum_{\substack{T\subseteq \underline{n}\\ T\subseteq \underline{n}}} (-1)^{|T|} \left|\bigcap_{i\in T} A_i\right|$$

$S=\emptyset$

$$\left|\bigcap_{i\in \underline{n}} A_i^c\right| = \sum_{k=0}^{n} (-1)^k \underbrace{\sum_{\substack{T\subseteq \underline{n}\\ |T|=k}} \left|\bigcap_{i\in T} A_i\right|}$$

|| DE MORGAN

$$\left|\Omega \setminus \bigcup_{i=1}^{n} A_i\right| = \sum_{k=0}^{n} (-1)^k S_k$$

PERCIÒ, FORMULA DI SYLVESTER:

$$\left|\Omega \setminus \bigcup_{i=1}^{n} A_i\right| = \sum_{k=0}^{n} (-1)^k S_k =$$

$(\ast)$

$$= \sum_{k=0}^{n} (-1)^k \sum_{\substack{T\subseteq \underline{n}\\ |T|=k}} \left|\bigcap_{i\in T} A_i\right|$$

CASO PARTICOLARE / ESPLICITO $\quad n=3$:

$$\left|\Omega \setminus A_1 \cup A_2 \cup A_3\right| =$$

$\underset{k=0}{\Omega} - \underset{k=1}{(|A_1| + |A_2| + |A_3|)} +$

$+ \underset{k=2}{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|} - \underset{k=3}{|A_1 \cap A_2 \cap A_3|}.$

FUNZIONE Φ DI EULERO :

DEF SIA $\Phi : \mathbb{Z}^+ \to \mathbb{Z}^+$ t.c.

$\forall n \in \mathbb{Z}^+$

$\Phi(n) = \# \{ m \in \mathbb{Z}^+ ; \underset{(m \leq n)}{m \in \underline{n}} \ \& \ \overset{m,n \text{ COPRIMI}}{MCD(m,n) = 1} \}$

AD ES : $\Phi(8) \overset{?}{=} 4$

$1, 2, 3, 4, 5, 6, 7, 8 \implies$

sì no sì no sì no sì no

TROVARE UNA FORMA CHIUSA PER Φ EULERO.

THM SIA $n \in \mathbb{Z}^+$. DAL TEOR. FOND. ARITMETICA SI HA:

$(*) \quad n \overset{!}{=} p_1^{i_1} p_2^{i_2} \cdots p_r^{i_r}$ $\quad p_1, \dots, p_r$ PRIMI A DUE DISTINTI.

AD ES : $20 = 2^2 \cdot 5$ $\quad p_1 = 2, p_2 = 5$

$30 = 2 \cdot 3 \cdot 5$ $\quad p_1 = 2, p_2 = 3, p_3 = 5$

DATA $(*)$ SI HA : (THM)

$$\to \Phi(n) = n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right) \quad ???$$

PROOF SIA $\Omega = \underline{n} = \{1, 2, \dots, n\}$

E DA $(*)$ SIANO (EVENTI PROIBITI)

$A_1 = \{ m \leq \underline{n} ; p_1 | m \}$

$A_2 = \{ m \leq \underline{n} ; p_2 | m \}$

$\tilde{A}_r = \{ m \leq n \ ; \ p_r | m \}.$ $\Omega = \underline{n}$

$$\phi(n) = \# \left| \underline{n} \setminus A_1 \cup A_2 \cup \dots \cup A_r \right| \underset{\neq}{\overset{\text{SYLVESTER}}{=}}$$

$$= \sum_{k=0}^{r} \sum_{\substack{T \subseteq \underline{r} \\ |T| = k}} \left| \bigcap_{i \in T} A_i \right| .$$ ← ORA CHI SONO QUESTI ???...

DATO $i = 1, 2, \dots, r$

i) $|A_i| = | \{ m \leq \underline{n} \ ; \ p_i | m \} | = \frac{n}{p_i}$

ii) $|A_{i_1} \cap A_{i_2}| = | \{ m \leq \underline{n} \ ; \ p_{i_1} | n \ \& \ p_{i_2} | n \} |$

$= | \{ m \leq \underline{n} \ ; \ p_{i_1} p_{i_2} | n \} | = \frac{n}{p_{i_1} p_{i_2}}$

iii) $|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = | \{ m \leq n ; \ p_{i_1} p_{i_2} p_{i_3} | m \} | = \frac{n}{p_{i_1} p_{i_2} p_{i_3}}$.

$n = 20 = 2^2 \cdot 5$, $p_1 = 2$, $p_2 = 5$

$|A_1| = 10$

$|A_2| = 4$

ECC.

QUINDI

$$\phi(n) = n \left(1 - \frac{1}{p_1} - \frac{1}{p_2} \dots - \frac{1}{p_r} + \sum_{\substack{i,j \\ i \neq j}} \frac{1}{p_i p_j} - \sum_{i,j,k} \frac{1}{p_i p_j p_k} + \dots \right) =$$

$$= n \left(1 - \frac{1}{p_1} \right) \left(1 - \frac{1}{p_2} \right) \dots \left(1 - \frac{1}{p_r} \right) .$$ C.V.D

AD ES $n \left(1 - \frac{1}{p_1} \right) \left(1 - \frac{1}{p_2} \right) =$

$n \left(1 - \frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{p_1 p_2} \right).$ OK.