Monday, October 16, 2023 11:26 AM

$(\mathbb{R}^n)^*$ SPAZIO DUALE DI $\mathbb{R}^n$:

$\left((\mathbb{R}^n)^* = \{ \varphi : \mathbb{R}^n \to \mathbb{R} \,;\, \varphi \text{ LINEARE} \}, + \,;\, \cdot \right)$ OVE

LO ZERO DI $(\mathbb{R}^n)^*$ È:

$\underline{0} : \mathbb{R}^n \to \mathbb{R}$ LINEARE, $\underline{0}(v) = 0 \in \mathbb{R}^n \quad \forall v \in \mathbb{R}^n$

$\underline{0}$ FUNZ IDENTICAMENTE NULLA.

FUNZIONI "COORDINATA": $i = 1, 2, \dots, n$

$dx_i : \mathbb{R}^n \to \mathbb{R}$ LINEARE t.c.

$$dx_i(\underline{e}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \overset{\text{NOT}}{=} \delta_{ij} \qquad j = 1, 2, \dots, n$$

DELTA DI KRONECKER.

QUINDI $dx_i \in (\mathbb{R}^n)^*$, $\forall i = 1, 2, \dots, n$.

ORA $\forall v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

$$\overset{!}{=} \sum_{j=1}^{n} v_j \cdot \underline{e}_j \implies$$

$$dx_i(v) = dx_i\left(\sum_{j=1}^{n} v_j \cdot \underline{e}_j \right) \overset{LIN}{=}$$

$$= \sum_{j=1}^{n} v_j \, \underbrace{dx_i(\underline{e}_j)}_{\delta_{ij}} = v_i \qquad !!!$$

THM L'INSIEME

$\{ dx_1, dx_2, \dots, dx_n \}$ È BASE DI $(\mathbb{R}^n)^*$!!

DA CUI $\dim((\mathbb{R}^n)^*) = \dim(\mathbb{R}^n) = n$.

PROOF

1) SISTEMA DI GENERATORI. SIA $\varphi \in (\mathbb{R}^n)^*$

SIHA SIA $v = (v_1, \dots, v_n) \in \mathbb{R}^n$

$$(*) \quad \boxed{\varphi \overset{!}{=} \sum_{i=1}^{n} \varphi(\underline{e}_i) \, dx_i \in (\mathbb{R}^n)^*} \qquad = \sum_{j=1}^{n} v_j \cdot \underline{e}_j$$

CONSEGUENZE 1) $\varphi : \mathbb{R}^n \to \mathbb{R}$ LINEARE;

ORA

COME SI SCRIVE φ ESPLICITAMENTE COME FUNZIONE

$$\varphi(v) = \varphi\left(\sum_{j=1}^{n} v_j \cdot \underline{e}_j \right) \overset{LIN}{=} \quad ???$$

$$\varphi = \varphi(x_1, x_2, \dots, x_n)$$

$$= \sum_{j=1}^{n} v_j \, \varphi(\underline{e}_j) \qquad \forall v \in \mathbb{R}^n$$

DATO $i = 1, 2, \dots, n$, COME SI SCRIVE

$$dx_i \varphi = \sum_{j=1}^{n} \varphi(\underline{e}_j) \, dx_i \quad \text{in } (\mathbb{R}^n)^*.$$

SIHA ~~$dx_i(x_1, x_2, \dots, x_n) = x_i$~~

QUINDI

$$\varphi(x_1, x_2, \dots, x_n) \overset{!}{=} \sum^{n} \varphi(\underline{e}_i) \, dx_i(x_1, x_2, \dots, x_n)$$

LINEARE CIOÈ: $\varphi : \mathbb{R}^n \to \mathbb{R}$ LINEARE $\iff \varphi \in (\mathbb{R}^n)^*$

SINTESI: SIA $v = (v_1, \dots, v_n) \in \mathbb{R}^n$

$$(*) \quad \boxed{\varphi \overset{!}{=} \sum_{i=1}^{n} \varphi(e_i)\, dx_i \in (\mathbb{R}^n)^*}$$

$$v = \sum_{j=1}^{n} v_j \cdot e_j$$

CONSEGUENZE 1) $\varphi : \mathbb{R}^n \xrightarrow{L} \mathbb{R}$ LINEARE;

ORA COME SI SCRIVE φ ESPLICITAMENTE COME FUNZIONE?

$$\varphi(v) = \varphi\left(\sum_{i=1}^{n} v_i \cdot e_i\right) \overset{LIN}{=} \sum_{i=1}^{n} v_i\, \varphi(e_i) \quad \forall v \in \mathbb{R}^n$$

$$\varphi = \varphi(x_1, x_2, \dots, x_n)\ ???$$

DATO $i \in \{1, 2, \dots, n\}$, COME SI SCRIVE dx_i?

$$dx_i \varphi = dx_i \sum_{j=1}^{n} \varphi(e_j)\, dx_j \quad \text{in } (\mathbb{R}^n)^*$$

SIMA ~~$dx_i(x_1, x_2, \dots, x_n) = x_i$~~

QUINDI

$$\varphi(x_1, x_2, \dots, x_n) \overset{!}{=} \sum_{i=1}^{n} \varphi(e_i)\, dx_i(x_1, x_2, \dots, x_n)$$

LINEARE CIOÈ: $\varphi : \mathbb{R}^n \to \mathbb{R}$ LINEARE $\iff \varphi \in (\mathbb{R}^n)^*$

$$\overset{!}{=} \sum_{i=1}^{n} \varphi(e_i) \cdot x_i$$

CIOÈ LINEARE, COEFF. $= \varphi(e_i)$

$$\varphi(x_1, \dots, x_n) \overset{!}{=} \sum c_i x_i$$

CIOÈ $\varphi : \mathbb{R}^n \to \mathbb{R}$ LINEARE $\iff$

φ SI SCRIVE COME POLINOMIO

OMOGENEO DI GRADO 1 (QUINDI SENZA TERMINE NOTO!)

AD ES. IN $n = 3$

i) $\varphi(x_1, x_2, x_3) = a_1 x_1 + a_2 x_2 + a_3 x_3$ LINEARE

ii) $f(x_1, x_2, x_3) = 2x_1 + 3x_2 - x_3 + 1$ NON LINEARE

iii) $h(x_1, x_2, x_3) = x_1 x_3 + x_2 - x_1 x_2$ NON LINEARE.

CASO PART., PER $n = 1$

$\exists\, \varphi : \mathbb{R} \to \mathbb{R}$ LINEARE $\iff$

$$\varphi(x_1) = c_1 x_1 \quad \text{(LO SAPEVAMO GIÀ ...)}$$

$$\varphi : \mathbb{R} \to \mathbb{R} \quad \frac{\partial f}{\partial e_i} \quad \varphi(x) = \varphi(x \cdot 1) = x\, \varphi(1) = c_1 x \quad x \in \mathbb{R}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx_i$$

CONSEGUENZE $df(x)$ È $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx_i$ DIFFERENZIALE

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A APERTO CONNESSO

$df(x)$ f DIFFERENZIABILE $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx_i$ $\forall f$

$L_x = df(x) : \mathbb{R}^n \to \mathbb{R}$ LINEARE, $df(x) \in (\mathbb{R}^n)^*$,

ES. $f : \mathbb{R}^3 \to \mathbb{R}$ DIFFERENZIABILE, $f(x, y, z) = 3x^2 y - xy^2 + 3z$

f POLINOMIALE $\Rightarrow$ f DIFFERENZIABILE),

$0 \;\overset{x}{=}\; \varphi : \mathbb{R} \to \mathbb{R}$ LINEARE $\overset{x}{\Longleftrightarrow}$

$\varphi(x_1) = c_1 x_1$ (LO SAPEVAMO GIÀ ...)

$\varphi : \mathbb{R} \to \mathbb{R}$ $\dfrac{\partial f}{\partial e_i}$ LINEARE $\varphi(x) = \varphi(x \cdot 1)$ $x \in \mathbb{R}$

$= x\,\varphi(1) = c_1 x$ ($\varphi(1) = c_1$ COST.)

$= \sum_{i=1}^{n} \dfrac{\partial f}{\partial x_i}(\underline{x})\, dx_i$

CONSEGUENZE $df(\underline{x})$ È CALCOLO $\sum_{i=1}^{n} \dfrac{\partial f}{\partial x_i}(\underline{x})\, dx_i$ DIFFERENZIALE

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A APERTO CONNESSO (?)

$df(\underline{x})$ f DIFFERENZIABILE $\sum_{i=1}^{n} \dfrac{\partial f}{\partial x_i}(\underline{x})$ ... $\forall f$!!

$L_x = df(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$ LINEARE, $df(\underline{x}) \in (\mathbb{R}^n)^*$,

ES. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = 3x^2 y - x y^2 + 3z$ DIFFERENZIALE DI f IN $\underline{x} \in A$.

f ... DIVENTA: IL TEOREMA (... DIFFERENZIABILE),

$\underline{x} = (1, -1, 1) \in \mathbb{R}^3$ ... È $\in (\mathbb{R}^n)^*$

$df(\underline{x})$! SI HA

$\dfrac{\partial f}{\partial x}(x, y, z) = 6xy - y^2 \Rightarrow \dfrac{\partial f}{\partial x}(1, -1, 1) = -6 - 1 = -7$

$\dfrac{\partial f}{\partial y}(x, y, z) = 3x^2 - 2xy \Rightarrow \dfrac{\partial f}{\partial y}(1, -1, 1) = 3 + 2 = 5$

$\dfrac{\partial f}{\partial z}(x, y, z) = 3 \Rightarrow \dfrac{\partial f}{\partial z}(1, -1, 1) = 3$ $(-7, 5, 3)$

$df(\underline{x}) \overset{!}{=} -7\,dx + 5\,y + 3\,z$