Monday, November 06, 2023 11:13 AM

WE RECALL.

(X, d) METRIC SPACE

1) GIVEN $x \in X$, $r \in \mathbb{R}^+$

$$\underset{\text{CENTER}}{I}(x, r) \overset{\text{DEF}}{=} \{ x' \in X \,;\, d(x, x') < r \}$$

CENTER RADIUS

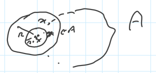
2) $A \subseteq X$, A OPEN $\overset{\text{DEF}}{\iff}$ $\forall x \in A$ $\exists r \in \mathbb{R}^+$ SUCH THAT

$$I(x, r) \subseteq A.$$

3) ACCUMULATION POINT $x_0 \in X$, $A \subseteq X$.

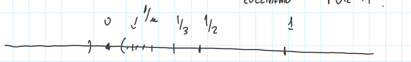
x_0 ACCUMULATION POINT FOR $A \subseteq X$ $\overset{\text{DEF}}{\iff}$

$$\forall r \in \mathbb{R}^+ \quad (I(x_0, r) - \{x_0\}) \cap A \neq \emptyset$$



EXAMPLE 1 LET $A = \{ 1/n \,;\, n \in \mathbb{Z}^+ \} \subseteq \mathbb{R}$. ARE THERE ACC. POINTS FOR A?

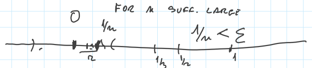
EUCLIDEAN



$\mathbb{R}$ EUCLIDEAN

LET $x_0 = 0$ IS AN ACC. POINT FOR A

FOR n SUFF. LARGE



$$I(x_0, r) = I(0, r) \subseteq]-1/2, 1/2[$$

SO $(I(0, r) - \{0\}) \cap A \neq \emptyset$

$\forall r \in \mathbb{R}^+$

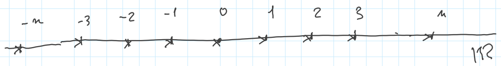
$\Rightarrow$ 0 ACC. POINT FOR $A = \{ 1/n \,;\, n \in \mathbb{Z}^+ \}$. YES

NOTICE THAT i) 0 ACC. POINT FOR A

ii) $0 \notin A$.

COUNTEREXAMPLE IN $\mathbb{R}$ EUCLIDEAN.

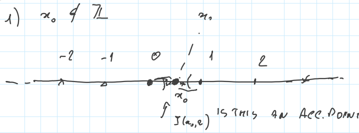
LET $A = \mathbb{Z}$



QUESTION: ARE THERE ACC. POINTS FOR $A = \mathbb{Z}$ IN $\mathbb{R}$ EUCLIDEAN.

TWO CASES

1) $x_0 \notin \mathbb{Z}$

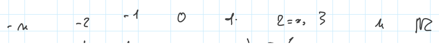


IS THIS AN ACC. POINT FOR $\mathbb{Z}$.

BUT IF $r \leq d(x_0, 0)$, $r \leq d(x, 1)$

THEN $I(x_0, r) \cap \mathbb{Z} = \emptyset \Rightarrow x_0$ NOT ACC. POINT

2) $x_0 \in \mathbb{Z}$ IS IT ACC. POINT FOR $\mathbb{Z}$?



SAY $x_0 = 2$ IS THIS AN ACC. POINT FOR $\mathbb{Z}$?

$I(2,r) =$

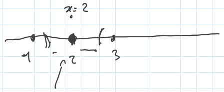
$=]2-r, 2+r[$

QUESTION IS: IS IT TRUE THAT

$\forall r \in \mathbb{R}^+ \quad (I(x_0, r) \setminus \{x_0\}) \cap \mathbb{Z} \neq \emptyset$? IS IT TRUE?

IF $r \leq 1$

$\Downarrow$

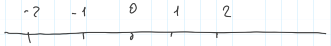


$I(2,r) \cap \mathbb{Z} = \{2\}$

$\Downarrow$

FOR $r \leq 1$

$(I(2,r) \setminus \{2\}) \cap \mathbb{Z} = \emptyset$ SO $x_0 \in \mathbb{Z}$ NOT ACC. POINT FOR $\mathbb{Z}$.

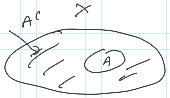


4) CLOSED SETS (X, d) METRIC SPACE

$A \subseteq X$

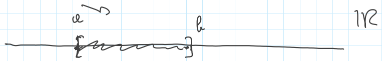
A CLOSED $\overset{DEF}{\Longleftrightarrow}$ $A^c \overset{DEF}{=} X \setminus A$ IS OPEN

COMPLEMENTARY SET

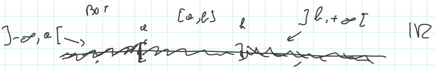


EX IN $\mathbb{R}$ EUCLIDEAN. LET $a, b \in \mathbb{R}$, $a < b$,

$[a,b] \overset{DEF}{=} \{x \in \mathbb{R} ; a \leq x \leq b\}$



IS ??? $[a,b]$ CLOSED $\overset{DEF}{\Longleftrightarrow}$ $[a,b]^c = \mathbb{R} \setminus [a,b]$ IS OPEN ??



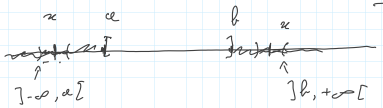
$[a,b]^c \overset{DEF}{=} \mathbb{R} \setminus [a,b]$

$=]-\infty, a[\ \dot{\cup} \ ]b, +\infty[$

$\{x \in \mathbb{R} ; x < a\}$ $\{x \in \mathbb{R} ; x > b\}$

IS $[a,b]^c =]-\infty, a[\ \dot{\cup} \ ]b, +\infty[$ OPEN ?

$\mathbb{R}$ YES!!!



$\Downarrow$

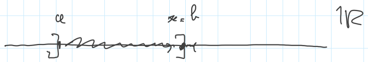
$[a,b]$ CLOSED SET !!!

$]a,b[$ OPEN

FURTHER EXAMPLE IN $\mathbb{R}$ EUCLIDEAN, $a, b \in \mathbb{R}$, $a < b$

LET

$$]a,b] \overset{\text{DEF}}{=} \{x \in \mathbb{R} \,;\, a < x \leq b\}$$

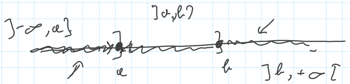


QUESTIONS:

1) $]a,b]$ OPEN? NO

NO, IT FAILS FOR $x = b \in]a,b]$.

2) $]a,b]$ CLOSED? $\overset{\text{DEF}}{\iff}$ $]a,b]^c = \mathbb{R} \setminus]a,b]$ OPEN??



$$]a,b]^c = \\ =]-\infty,a] \,\dot\cup\,]b,+\infty[$$

$$\{x \in \mathbb{R} \,;\, x \overset{!}{\leq} a\} \;\dot\cup\; \{x \in \mathbb{R} \,;\, x > b\} \text{ IS OPEN}$$

SET $x = a$ THE COND TO BE OPEN FAILS HERE!!!

$\Downarrow$

$$]a,b]^c =]-\infty,a] \,\dot\cup\,]b,+\infty[\quad \text{NOT OPEN}$$

$\Updownarrow$

$]a,b]$ IS NOT CLOSED !!!

NOTICE THAT

$]\overset{\circ}{a},b]$ IS NEITHER OPEN NOR CLOSED !!!

$\iff$

S ...

AT 10.15