Monday, November 06, 2023 11:13 AM

(X, d) METRIC SPACE

DEF (X, d) M.S. IS DISCONNECTED $\overset{DEF}{\iff}$

$\exists A_1, A_2 \subseteq X$, $A_1, A_2 \neq \emptyset, X$ SUCH THAT

i) A_1, A_2 ARE OPEN

ii) $\underbrace{A_1 \cap A_2 = \emptyset}_{+}$

$$A_1 \dot{\cup} A_2 = X$$

EX $\mathbb{R} \setminus \{0\} = \underbrace{\overset{\neq \emptyset, \mathbb{R} \setminus \{0\}}{]-\infty, 0[}}_{\text{OPEN}} \dot{\cup} \underbrace{\overset{\neq \emptyset, \mathbb{R} \setminus \{0\}}{]0, +\infty[}}_{\text{OPEN}} \implies X = \mathbb{R} \setminus \{0\}$ DISCONNECTED

CLEARLY, (X, d) CONNECTED $\overset{DEF}{\iff}$

(X, d) IS NOT DISCONNECTED.

PROP 1 (X,d) METRIC SPACE.

(X,d) DISCONNECTED $\overset{!!!}{\Longleftrightarrow}$ $\exists\, A \neq \emptyset, X$ WHERE

A OPEN / CLOSED.

COROLLARY

(X,d) CONNECTED $\Longleftrightarrow$ GIVEN $A \subseteq X$, $A \neq \emptyset, X$

THEN

i) A OPEN $\Longrightarrow$ A NOT CLOSED

ii) A CLOSED $\Longrightarrow$ A NOT OPEN

PROOF OF PROP. 1

$\Longrightarrow$) (X,d) M.S. DISCONNECTED $\overset{\text{DEF}}{\Longleftrightarrow}$

$\exists\, A_1, A_2 \subseteq X$, $A_1, A_2 \neq \emptyset, X$ A_1, A_2 OPEN

SUCH THAT

$$X = A_1 \dot{\cup} A_2 \quad \left(\Longleftrightarrow \begin{cases} A_1^c = A_2 \\ A_2^c = A_1 \end{cases} \right)$$

BUT NOW, SET

$$A = A_1 \quad \underline{\underline{\text{OPEN}}}$$

BUT $A^c = (A_1)^c = A_2$ OPEN $\Longrightarrow$ A_1 $\underline{\underline{\text{CLOSED}}}$

$A = A_1$ BOTH $\underline{\underline{\text{OPEN}}}$ / $\underline{\underline{\text{CLOSED}}}$

$$\boxed{\begin{array}{l}\underline{\text{N.B}} \\ A = A_1 \neq \phi, X\end{array}}$$

$\Longleftarrow$) $\exists A \subseteq X$, $\underline{A \neq \phi, X}$

A $\underline{\text{OPEN}}$ / $\underline{\underline{\text{CLOSED}}}$

SET $A_1 = A$, SET $A_2 = A^c$

$A = A_1$ OPEN, $A_2 = (A_1)^c \Longrightarrow (A^c)^c = A$ $\underline{\text{CLOSED}}$

$$\Downarrow$$

A_2 OPEN

SO $X = A_1 \dot{\cup} A_2 = A \dot{\cup} A^c \Longrightarrow X$

DISCONNECTED

OPEN OPEN OPEN OPEN

AS DESIRED.

THM (EUCLIDEAN CONNECTION THM)

$n \in \mathbb{Z}^+$

THE SPACES $\mathbb{R}^n$ EUCLIDEAN ARE CONNECTED SPACES !!!

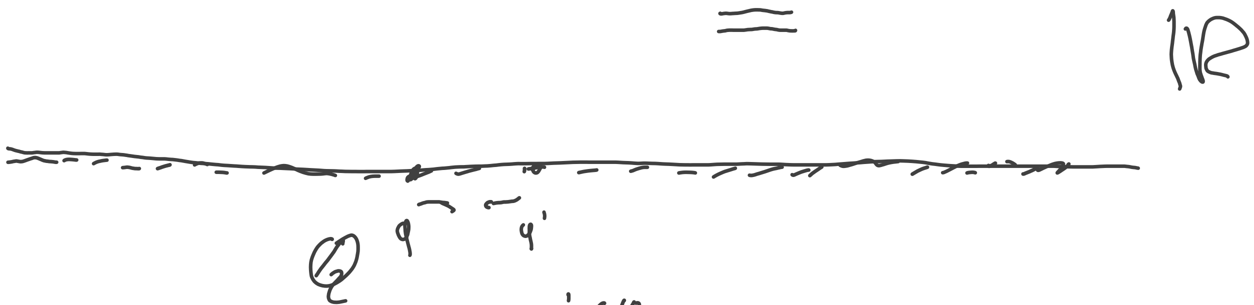
TO FIX IDEAS

IF $n=1$, $\mathbb{R} = \mathbb{R}^1$ IS CONNECTED.

CONSIDER THE SUBSPACE $\mathbb{Q} \subseteq \mathbb{R}$

$\mathbb{Q}$ ← RATIONALS

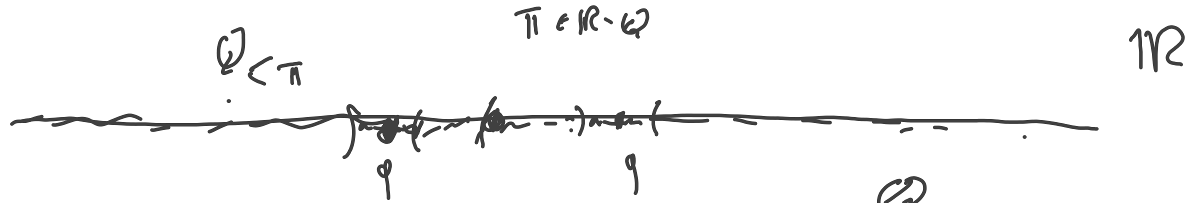
IS $\mathbb{Q}$ CONNECTED ??? NO



$q, q' \in \mathbb{Q}$

WHY $\mathbb{Q}$ NOT CONNECTED ???...

FIX $\pi \in \mathbb{R} \setminus \mathbb{Q}$



SET $\mathbb{Q}_{<\pi} \overset{\text{DEF}}{=} \{ q \in \mathbb{Q} \; ; \; q < \pi \}^{\mathbb{Q}} \subseteq \mathbb{Q}$

$$\mathbb{Q}_{>\pi} \overset{\text{DEF}}{=} \{ q \in \mathbb{Q} \; ; \; q > \pi \}$$

NOTICE THAT

$$\mathbb{Q}_{<\pi}, \mathbb{Q}_{>\pi} \neq \emptyset, \mathbb{Q}$$

AND

$$\mathbb{Q}_{<\pi} \cap \mathbb{Q}_{>\pi} = \emptyset \;\; \& \;\; \mathbb{Q}_{<\pi} \,\dot{\cup}\, \mathbb{Q}_{>\pi} = \mathbb{Q}$$

BUT $\mathbb{Q}_{<\pi}$, $\mathbb{Q}_{>\pi}$ ARE **OPEN** IN $\mathbb{Q}$.

SO $\quad Q = Q_{<\pi} \dot{\cup} Q_{>\pi}$ $\quad Q_{<\pi}, Q_{>\pi}$ NON TRIVIAL

OPEN $\quad$ OPEN