Tuesday, November 14, 2023   7:36 AM

$(X, \|\;\|)$ NORMED SPACE

VECTOR SPACE — NORM

$\|\;\| : X \longrightarrow \mathbb{R}$ SUCH THAT

i) $\|x\| \geq 0$, $\|x\| = 0 \iff x = \underline{0}$

ii) $\lambda \in \mathbb{R}$, $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$

iii) $\|x + y\| \leq \|x\| + \|y\|$ TRIANG. INEQ FOR NORMS

$(X, \|\;\|)$ NORMED $\Rightarrow$ (X, d) METRIC SPACE

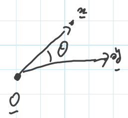
WHERE

$$d(\underline{x}, \underline{y}) \overset{\text{DEF}}{=} \|\underline{x} - \underline{y}\|$$

$$\Downarrow$$

$$\|x\| = d(\underline{x}, \underline{0}) \;\; ??? \, .$$

ANGLES??? THEY ARE NOT DEFINED IN NORMED SPACES !!??



???

SPACES WITH INNER PRODUCT

$(X, \langle\,,\rangle)$, $\langle\,,\rangle : X \times X \longrightarrow \mathbb{R}$

VECTOR SPACE — INNER PRODUCT

SUCH THAT

i) $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$

&

$\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle$

ii) $\lambda \in \mathbb{R}$ $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle = \langle x, \lambda y\rangle$

???

iii) $\langle x, y\rangle = \langle y, x\rangle$ SYMMETRY

iv) $\langle x, x\rangle \geq 0$ & $\langle x, x\rangle = 0 \iff x = \underline{0}$

POSITIVELY DEFINED QUADRATIC FORM.

LINEAR FUNCTS

$F : X \to Y$, X, Y VECTOR SPACES

F LINEAR $\iff \begin{cases} F(x_1 + x_2) = F(x_1) + F(x_2) & \text{ADDITIVITY} \\ F(\lambda x) = \lambda \cdot F(x) & \text{HOMOGENEITY} \\ \lambda \in \mathbb{R} \end{cases}$

i), ii) BILINEAR !!!

EX $X = \mathbb{R}^n = \{ x = (x_1, \dots, x_n) ;\ x_i \in \mathbb{R},\ i = 1, 2, \dots, n \}$

GIVEN $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

SET

$$(*) \quad \langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

THIS SATISFIES i), ii), iii), iv)

(*) IS AN INNER PRODUCT ON $\mathbb{R}^n$

CALLED EUCLIDEAN INNER PRODUCT.

IT IS EASY TO SHOW THAT

$\forall x \in X \qquad \|x\| = \sqrt{\langle x, x \rangle}$ IS A NORM !!!

$(X, \langle , \rangle)$ INNER PRODUCT SPACE $\implies$ IT IS A NORMED SPACE

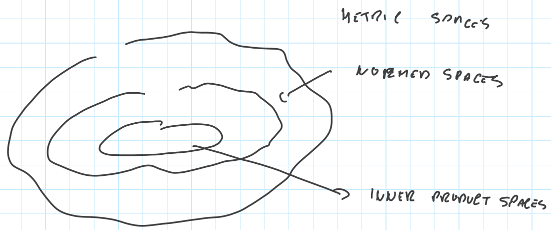
BY SETTING

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$\implies$ IT IS ALSO METRIC SPACE

BY SETTING

$$\boxed{d(x, y) \overset{\text{def}}{=} \|x - y\| = \sqrt{\langle x - y, x - y \rangle}}$$



INNER PRODUCT SPACES AND ANGLES ...

INNER PRODUCT SPACES AND ANGLES

THM (CAUCHY/SCHWARTZ INEQUALITY)

$\underline{x}, \underline{y} \in (X, \langle , \rangle)$. THEN

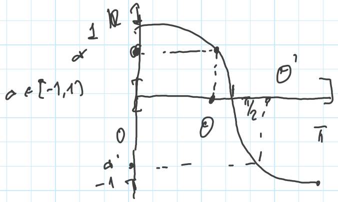
$$(*) \quad |\langle \underline{x}, \underline{y} \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle} = \|\underline{x}\| \cdot \|\underline{y}\|$$

COROLLARY $\underline{x}, \underline{y} \neq \underline{0}$

THEN $(*) \Rightarrow \dfrac{|\langle \underline{x}, \underline{y} \rangle|}{\|\underline{x}\| \cdot \|\underline{y}\|} \leq 1 \iff$

$$(**) \rightsquigarrow \boxed{-1 \leq \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{x}\| \cdot \|\underline{y}\|} \leq 1}$$

RECALL THE COS FUNCT.



$\cos : [0, \pi] \longrightarrow \mathbb{R}$

$\cos : [0, \pi] \underset{1-1}{\overset{SO}{\longrightarrow}} [-1, 1]$

$\mathbb{R} \; \exists! \, \theta \in [0, \pi]$ S.T.

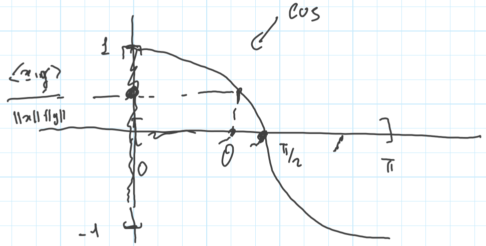
$\cos \theta = \alpha$

$(X, \langle , \rangle)$ INNER PRODUCT SPACE

$\underline{x}, \underline{y} \neq 0$

$\Downarrow$

$-1 \leq \dfrac{\langle x, y \rangle}{\|\underline{x}\| \|\underline{y}\|} \leq 1$

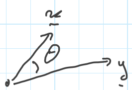


$\exists! \; \theta \in [0, \pi]$ SUCH THAT

$$\cos \theta = \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{x}\| \cdot \|\underline{y}\|}$$

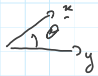
SO θ IS THE ANGLE BETWEEN THE NONZERO VECTORS

$\underline{x}, \underline{y} \neq \underline{0}$.

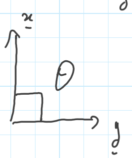


SPECIAL CASES

1) $\langle \underline{x}, \underline{y} \rangle > 0 \iff 0 \le \theta < \pi/2 \iff \theta$ ACUTE



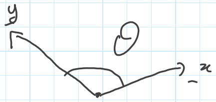
2) $\langle \underline{x}, \underline{y} \rangle = 0 \iff \theta = \pi/2 \iff$



$\underline{x}, \underline{y}$ PERPENDICULAR

$\underline{x}, \underline{y}$ ORTHOGONAL

3) $\langle \underline{x}, \underline{y} \rangle < 0 \iff \pi/2 < \theta \le \pi \iff$



BREAK QUESTIONS?

BEGIN AGAIN AT 10.20