Wednesday, November 15, 2023 12:03 PM

WE RECALL:

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

FIX A DIRECTION $v \in \mathbb{R}^n$, i.e. $\|v\| = 1$.

f ADMITS DIRECTIONAL DERIVATIVE AT $\underline{x} \in A$

ALONG THE DIRECTION v: $\|v\| = 1$

IF AND ONLY IF

DIR. DER.

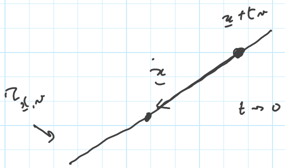
(*) $\exists$ FINITE $\displaystyle\lim_{t \to 0} \frac{f(\underline{x} + tv) - f(\underline{x})}{t} \left(\overset{\text{NOT}}{=} \frac{\partial f}{\partial v}(\underline{x}) \in \mathbb{R} \right)$

RECALL $r_{\underline{x}, v} = \{ \underline{x} + tv ; \ t \in \mathbb{R} \}$

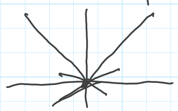
IS THE UNIQUE LINE PASSING THROUGH $\underline{x} \in A$

&

HAVING DIRECTION v



$f(x) = |x|$



WE HAVE: $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$. $(n > 1)$

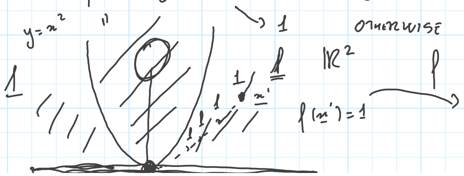
EVEN f ADMITS ALL DIRECTIONAL DERIVATIVES

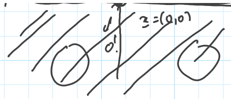
AT $\underline{x} \in A$ $\not\Rightarrow$ f IS CONTINUOUS AT $\underline{x} \in A$

COUNTEREXAMPLE $n = 2$.

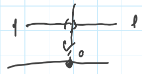
LET $f: \mathbb{R}^2 \to \mathbb{R}$ SUCH THAT

$$f(x, y) = \begin{cases} 0 & y \geq x^2 \quad 1) \\ 0 & y \leq 0 \quad 2) \\ 1 & \text{OTHERWISE} \quad 3) \end{cases}$$





LET $\underline{x} = (0,0)$ IR



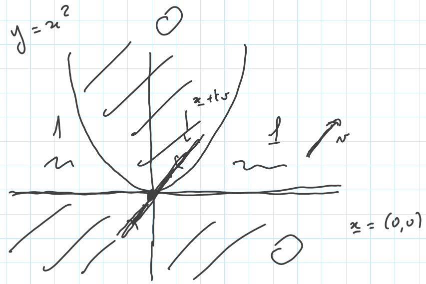
CLEARLY, $\underline{f(\underline{x}) = f(0,0) = 0}$!!!

BUT, IS f CONT. AT $\underline{x} = (0,0)$??? $\underline{\underline{\text{NO!!!}}}$

BUT, GIVEN $v : \|v\| = 1$,

WHAT ABOUT

? $\dfrac{\partial f}{\partial v}(\underline{x})$?

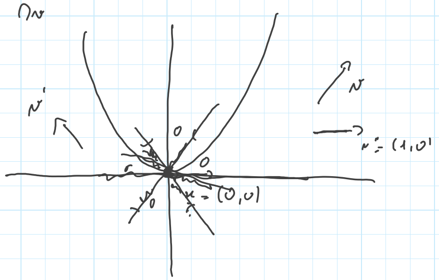


$\underline{\exists \text{FINITE}}$? $\dfrac{\partial f}{\partial v}(\underline{x}) = \lim\limits_{t \to 0} \dfrac{f(\underline{x}+tv) - f(\underline{x})}{t}$???

$= \lim\limits_{t \to 0} \dfrac{0 - 0}{t} = 0$!!!

THEN, WE PROVED:

$\exists \; \dfrac{\partial f}{\partial v}(\underline{x}) = 0$



$\underline{\text{DIFFERENTIABLE FUNCTIONS}}$

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$. $(\underline{n \geq 1})$

DEF f DIFFERENTIABLE AT $x \in A$

DIFFERENTIAL OF f AT $x \in A$ IF AND ONLY IF

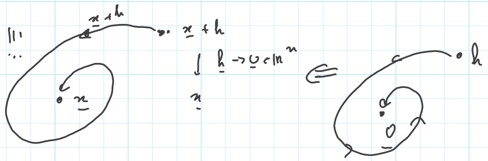
$\exists\ L_x : \mathbb{R}^n \to \mathbb{R}$ LINEAR $\nearrow \begin{cases} L_x(h_1+h_2) = L_x(h_1)+L_x(h_2) \\ L_x(\lambda \cdot h) = \lambda \cdot L_x(h) \end{cases}$

SUCH THAT

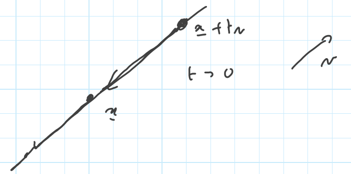
$$\boxed{\lim_{\substack{h \to 0 \in \mathbb{R}^n}} \frac{f(x+h) - f(x) - L_x(h)}{\|h\|} = 0} \qquad (*) \ ???$$

$\mathbb{R}^n$

GENERAL !!! $\Rightarrow$



COMPARE



FIRST "EXPLANATION" OF $(*)$ BY MEANS OF "APPROXIMATION THEORY"

???

$$E_x(h) = f(x+h) - f(x) - L_x(h) =$$

THE UPPER PART OF THE RATIO

"ERROR"

$$= f(x+h) - \big(f(x) + L_x(h)\big)$$

TRUE VALUE — COST — LINEAR FUNCT

IS THE "ERROR" THAT WE MAKE WHEN WE REPLACE $f(x+h)$

WITH

$$f(x) + L_x(h)$$

COST — LINEAR

AND $(*)$ (f DIFFERENTIABLE) $\Longleftrightarrow$

$$\Longleftrightarrow \lim \frac{E_x(h)}{\|h\|} = 0 \quad \left(\frac{E_x(h)}{\ldots} \xrightarrow[h \to \ldots]{\text{STRONGER}} 0 \right) \ (**)$$

$\underline{h} \to \underline{0} \in \mathbb{R}^n$ $\qquad \|\underline{h}\|$ $\qquad \|\underline{h}\| \quad \underline{h} \to \underline{0} \in \mathbb{R}^n$

NOTICE THAT $\underline{h} \to \underline{0} \iff \|\underline{h}\| \xrightarrow[\underline{h}\to\underline{0}]{} 0$

$\Downarrow \quad \not\Uparrow$

$E_{\underline{x}}(\underline{h}) \xrightarrow[\underline{h}\to\underline{0}]{} 0$

BREAK $\qquad$ QUESTIONS?

BEGIN AGAIN AT $\sim$ 15.10