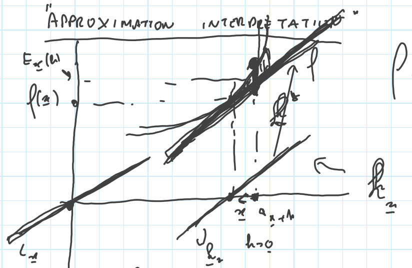
Wednesday, November 15, 2023 12:04 PM

"GEOMETRIC INTERPRETATION" OF THE PREVIOUS "APPROXIMATION INTERPRETATION" (FOR $n=1$)



$f: A \subseteq \mathbb{R} \to \mathbb{R}$

$L_x: \mathbb{R} \to \mathbb{R}$ LINEAR

$L_x \to \ell_x$

NOW, SET $\ell_x(x+h) = L_x(h)$?

AND SET

$$\ell^*_x(x+h) = \ell_x(x+h) + \underbrace{f(x)}_{\text{cost}}$$

BUT

$$\to E_x(h) = f(x+h) - (f(x) + L_x(h)) \Leftarrow$$

$$= f(x+h) - \ell^*_x(x+h)$$

AND (BY DIFF)

$$\frac{E_x(h)}{\|h\|} \xrightarrow[h \to 0]{} 0 \quad !!!$$

WE HAVE

PROP $\boxed{\text{IF } n = 1}$

f DIFFERENTIABLE AT $x \in A \iff f$ ADMITS DERIVATIVE ($f'(x) \in \mathbb{R}$) AT $x \in A$

PROOF $\Rightarrow$)

$f: A \subseteq \mathbb{R}^1 \to \mathbb{R}$ DIFF AT $x \in A \overset{\text{DEF}}{\iff}$

$\exists\, L_x: \mathbb{R} \to \mathbb{R}$ LINEAR SUCH THAT

$f(x+h) - f(x) - L_x(h)$

$$(\dagger)\quad \lim_{h\to 0\in\mathbb{R}} \frac{f(x+h)-f(x)-L_x(h)}{|h|} = 0$$

$h \in \mathbb{R}$

$\|h\| = |h|$

CONSIDER

$$(\dagger):\quad \lim_{h\to 0\in\mathbb{R}} \frac{f(\underline{x}+h)-f(\underline{x})-L_{\underline{x}}(h)}{|h|} \overset{!!}{=} 0$$

AND

$$(\dagger\dagger)\quad \lim_{h\to 0\in\mathbb{R}} \frac{f(\underline{x}+h)-f(\underline{x})-L_{\underline{x}}(h)}{h} \overset{!!}{=} 0$$

SINCE BOTH $(\dagger)$ AND $(\dagger\dagger)$ ARE ASSUMED TO ZERO

$\Downarrow$

$(\dagger)$ AND $(\dagger\dagger)$ ARE EQUIVALENT !!

$\Rightarrow)$

HENCE :

f DIFF AT $\underline{x}\in A \iff \exists\, L_{\underline{x}}:\mathbb{R}\to\mathbb{R}$ LINEAR

S.T.

$$(\dagger\dagger)\quad \lim_{h\to 0} \frac{f(\underline{x}+h)-f(\underline{x})-L_{\underline{x}}(h)}{h} = 0$$

NOW

$$(\dagger\dagger) \iff \underline{\lim_{h\to 0} \frac{f(\underline{x}+h)-f(\underline{x})}{h} = \lim_{h\to 0}} \frac{L_{\underline{x}}(h)}{h} = L_{\underline{x}}(1)\ (\dagger\dagger\dagger)$$

WHAT IS $L_{\underline{x}}(h) =$???

$h \in \mathbb{R}$?

$h = h\cdot 1$

$$L_{\underline{x}}(h) = h\cdot L_{\underline{x}}(1)$$

$$\overset{\text{COST}}{=} \lim_{h\to 0} \frac{\cancel{h}\cdot L_{\underline{x}}(1)}{\cancel{h}} = L_{\underline{x}}(1) \in \mathbb{R}$$

$$(\dagger\dagger) \iff \underbrace{\lim_{h\to 0\in\mathbb{R}} \frac{f(\underline{x}+h)-f(\underline{x})}{h}}_{} = L_{\underline{x}}(1) \in \mathbb{R}$$

INCR. RATIO

$\Downarrow$ MEANS THAT

f ADMITS DERIVATIVE AT $\underline{x}\in A$ & $f'(\underline{x}) = L_{\underline{x}}(1)$!!

SO $\Rightarrow$) IS PROVED, AND MORE:

$$f'(\underline{x}) = L_{\underline{x}}(1) \quad !!!$$

QED

$\Leftarrow$) WHAT IS

$L_{\underline{x}} : \mathbb{R} \to \mathbb{R}$ LINEAR

WE CONJECTURE THAT:

$L_{\underline{x}} : \mathbb{R} \to \mathbb{R}$ LINEAR

IS

$$\Big(L_{\underline{x}}(h) \overset{DEF}{=} \underset{\underset{COST}{\parallel}}{f'(\underline{x})} \cdot h \qquad \forall h \in \mathbb{R} \Big) \quad (\text{ASSUMPTION})$$

f ADMITS DERIVATIVE $f'(\underline{x}) \in \mathbb{R}$ AT $\underline{x} \in A$

LET US CONSIDER.

$$\lim_{h \to 0} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{|h|} = 0 \quad ??? \text{ (IS IT TRUE???)}$$

YES

$$\Updownarrow$$

$$\lim_{h \to 0} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{h} = 0 \quad ??? \text{ (IS IT TRUE ???)}$$

YES

$$\Updownarrow$$

$$\lim_{h \to 0} \frac{f(\underline{x}+h) - f(\underline{x})}{h} \overset{?}{=} \lim_{h \to 0} \frac{L_{\underline{x}}(h)}{h} \quad ??? \text{ (IS IT TRUE ???)}$$

YES

BUT, BY OUR ASSUMPTION

$$\Leftarrow \Big(L_{\underline{x}}(h) = f'(\underline{x}) \cdot h \quad \forall h \in \mathbb{R}$$

$$\Updownarrow$$

$$\lim_{h \to 0} \frac{f(\underline{x}+h) - f(\underline{x})}{h} = \lim_{h \to 0} \frac{f'(\underline{x}) \cdot \cancel{h}}{\cancel{h}} = f'(\underline{x})$$

IS IT TRUE??? YES

YES, IT IS DEFINITION OF THE DERIVATIVE $f'(\underline{x})$!!!

SO, WE PROVED $\Leftarrow$), AND MORE

$$L_{\underline{x}}(h) = f'(\underline{x}) \cdot h. \quad !!!$$

IS SYNTHESE

$\Rightarrow)$, MORE $f'(x) = L_x(1)$

$\Leftarrow)$, MORE $L_x(h) = f'(x) \cdot h$, $h \in \mathbb{R}$

BREAK

QUESTIONS ?

BYEBYE