Wednesday, November 15, 2023 12:09 PM

PARTIAL DERIVATIVES

CANONICAL BASIS OF $\mathbb{R}^n$

CAN. BASIS IS $\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n\} \subseteq \mathbb{R}^n$

WHERE $i = 1, 2, \dots, n$

$\underline{e}_i = (0, 0, \dots, 0, \underset{i\text{-th}}{1}, 0, \dots, 0)$ ← ARE DIRECTIONS SINCE $\|\underline{e}_i\| = 1$

EX $n = 3$ $\quad \underline{e}_1 = (1,0,0), \ \underline{e}_2 = (0,1,0), \ \underline{e}_3 = (0,0,1)$

GIVEN $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

$$= \sum_{i=1}^{n} v_i \cdot \underline{e}_i \qquad (*)$$

EX BIS IN $\mathbb{R}^3$. LET $v = (2, -1, 3)$ THEN

$$v = (2, -1, 3) = 2 \cdot \underline{e}_1 - 1 \cdot \underline{e}_2 + 3 \underline{e}_3 =$$

$$= 2(1,0,0) - 1(0,1,0) + 3(0,0,1)$$

GIVEN $i = 1, 2, \dots, n$ THE i-th PARTIAL DERIVATION AT $\underline{x} \in A$

(OF $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$)

IS

$$\frac{\partial f}{\partial \underline{e}_i}(\underline{x}) \overset{\text{DEF}}{=} \overset{\text{FINITE}}{\lim_{t \to 0}} \frac{f(\underline{x} + t\underline{e}_i) - f(\underline{x})}{t} \qquad (\dagger)$$

REWRITE $(\dagger)$ IN TERMS OF "VARIABLES".

WRITE $\underline{x} = (x_1, x_2, \dots, x_n)$ $\quad t\underline{e}_i = (0, 0, \dots, 0, \underset{i\text{-th}}{t}, 0, \dots, 0)$

$(\dagger)$ BECOMES

$$\text{FINITE} \ \lim_{t \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{t} \ ??? \qquad (\ddagger)$$

TWO QUESTIONS

i) WHY "PARTIAL" ???

ii) WE WRITE (AS USUAL NOTATION)

$\boxed{\frac{\partial f}{\cdots}(\cdots)} \ \cdots \ \boxed{\frac{\partial f}{\cdots}(x)}$?

$\boxed{\frac{\partial f}{\partial x_i}(\underline{x})}$ IN PLACE $\boxed{\frac{\partial f}{\partial e_i}(\underline{x})}$.

FINITE, i-th, TRUE VARIABLE

$$(\#) \quad \lim_{t \to 0} \frac{f(x_1, \dots, x_i + t, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{t} \quad ???$$

EX $n = 3$ $\quad f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) \stackrel{DEF}{=} x^3 z + x y^2 z^3 - 3y^2 z$

$$\frac{\partial f}{\partial x}(x,y,z) = 3x^2 z + y^2 z^3$$

$$\frac{\partial f}{\partial y}(x,y,z) = 2xyz^3 - 6yz$$

$$\frac{\partial f}{\partial z}(x,y,z) = x^3 + 3xy^2z^2 - 3y^2$$

LET $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

THM IF f DIFFERENTIABLE AT $\underline{x} \in A \Longrightarrow$

$\Rightarrow \forall v : \|v\| = 1$ DIRECTION

$$\exists \frac{\partial f}{\partial v}(\underline{x}) \stackrel{!}{=} L_{\underline{x}}(v) \quad , \; v \text{ DIRECTION } \|v\| = 1.$$

PROOF LET f DIFF AT $\underline{x} \in A \stackrel{DEF}{\Longleftrightarrow}$

$\exists L_{\underline{x}} : \mathbb{R}^n \to \mathbb{R}$ LINEAR SUCH THAT

$$\lim_{h \to \underline{0} \in \mathbb{R}^n} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{\|h\|} = 0 \qquad (*)$$

BY "SPECIALIZATION" SETTING $h = t \cdot v$ $\quad t \in \mathbb{R}$

$(*)$ BECOMES

NOTICE $(*) \Rightarrow (**)$, $\not\Leftarrow$

$$\lim_{\substack{tv \to \underline{0} \\ t \to 0 \in \mathbb{R}}} \frac{f(\underline{x}+tv) - f(\underline{x}) - L_{\underline{x}}(tv)}{\|t \cdot v\|} = 0 \qquad (**)$$

BUT $\|t \cdot v\| = |t| \cdot \|v\| = |t|$

$$\overset{\|}{1}$$

(**) BECOMES

(***) $$\lim_{t \to 0 \in \mathbb{R}} \frac{f(\underline{x}+t\underline{v}) - f(\underline{x}) - L_{\underline{x}}(t\underline{v})}{|t|} = 0$$

$$\Updownarrow$$

$$\lim_{t \to 0 \in \mathbb{R}} \frac{f(\underline{x}+t\underline{v}) - f(\underline{x}) - L_{\underline{x}}(t\underline{v})}{t} = 0$$

$$L_{\underline{x}}(t\underline{v}) \overset{\text{LIN}}{=} t \cdot L_{\underline{x}}(\underline{v})$$

$$\Updownarrow$$

$$\lim_{t \to 0} \frac{f(\underline{x}+t\underline{v}) - f(\underline{x})}{t} = \lim_{t \to 0} \frac{L_{\underline{x}}(t\underline{v})}{t} = \lim_{t \to 0} \frac{\cancel{t} \cdot L_{\underline{x}}(\underline{v})}{\cancel{t}} = L_{\underline{x}}(\underline{v})$$

IN SYNTHES, WE PROVED THAT f DIFF AT $\underline{x} \in A$ IMPLIES

$$\exists \text{ FINITE } \lim_{t \to 0} \frac{f(\underline{x}+t\underline{v}) - f(\underline{x})}{t} = L_{\underline{x}}(\underline{v}) \in \mathbb{R}$$

THAT IS $$\boxed{\exists \frac{\partial f}{\partial \underline{v}}(\underline{x}) = L_{\underline{x}}(\underline{v})}$$

QED

BREAK QUESTIONS?

BEGIN AGAIN AT 10.10